MATHESIS ENUCLEATA:

OR, THE

ELEMENTS

OF THE

MATHEMATICKS.

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The AUTHOR's PREFACE TO THE READER,

Containing a SYNOPSIS of his Method.

I.

THAT the Reader may the better apprehend our design and aim, we have thought fit to premise some things concerning the Methods, both general and particular, we make use of in the following Treatise. For as heretofore a sort of a blind deference to, and superstitious Veneration of Antiquity, and especially of Aristotle, *has hindred the growth and progress of Natural Philosophy, which of late has made such considerable advances, since it has ventured to stand upon its own Bottom, to make new Additions to former Inventions, to essay new and unknown Objects, to substitute Things instead of Names, Certainties instead of Doubts, and Experience in the room of dull Credulities; not derogating in the mean while from the deserved Praises of the Ingenious among the Ancients: So without doubt Mathematicks also, unless our Predecessors had imagined that it had long ago been brought to its utmost Perfection by* Euclid, Archimedes, Apollonius, *and other Ingenious Ancients, would have arrived long since to a higher pitch, and by this time have surpass'd those Limits, which now we admire its arrival to.*

II.

It is confess'd by all, that no Human Knowledge whatsoever can lay a more just claim to an unshaken Evidence and Certainty, or boast an higher necessity of its Demonstrations, or a greater multitude of undeniable Truths, than the Mathematicks; and that those Propositions we have, found out by Archimedes, *demonstrated by* Euclid, Apollonius, *and others, are at the same time unquestionable and altogether wonderful. But we may with Truth affirm, that most of their Propositions may either be disposed in a better order, or propounded easier, or demonstrated more evidently and directly, or taught after a more short and compendious way, now at least after they are already found out, and with a great deal of Pains demonstrated by their first Inventors; and of this Opinion are several of the best and most celebrated Mathematicians of the present Age.*

III.

It is certain Euclid *has demonstrated several Propositions, (as* Prop. 2, 3, 20, 30, lib. 1. 2, 5, 6, 10, 15, 28, 29, lib. 3. *&c.) whose Truth to any attentive Person appears from the very terms, more clearly and certainly than the truth of* Axiom 13. lib. 1. *which his Interpreters dare not admit without a Demonstration. And tho' those superfluous Demonstrations derogate nothing from the certitude of the thing, yet by an unnecessary increase of the number of Propositions, and (which frequently follows thence) an inverting the order of things, they breed Tediousness and Confusion.*

IV.

There are none, unless those who are bigotted to Antiquity, but must own that the Elements of Euclid *are destitute of a just and orderly Disposition of things. For to omit, that in the first Book there are handled several sorts of Subjects, and a great variety of Properties demonstrated of them promiscuously without any respect to similitude or conveniency; there is this never to be*

excused,

excused, that, after he has in the first Book deduced and demonstrated some particular Affections of Magnitude, he proceeds, in the second, to those things which are universal and common to any quantity; then in the third and fourth, he contemplates the Circle and the Properties of Figures inscrib'd in it, or circumscribed about it; in the fifth again he treats of the universal Doctrine of Reasons and Proportions; and yet not so universally, but in the seventh again he is obliged to demonstrate the same of Numbers particularly, which might have been done for all Quantities whatsoever, by one general Demonstration.

V.

Then as for the method of Demonstrating used by the Ancients; it is true that it nicely regarded the certainty of its Conclusions, nor would it admit any thing into its Demonstrations, which was not either a first Principle and so self evident, (called by them an Axiom*) or might not be supposed, beyond all Contradiction, possible to be effected (and on that account named a* Postulate*) or, thirdly, an arbitrary Denomination of the thing proposed which needed no Demonstration, (and was called a Definition or Explication of a Term;) or, lastly, which had not been evidently demonstrated before: Yet I believe none will deny, but that this Method would have been more deservedly esteem'd, if with the certainty of its Conclusions it had joined a greater Easiness, Brevity, and Evidence, which is wanting in most of the Demonstrations of the Ancients; who thought it enough, firmly and infallibly to establish the truth of their Theorems, and extort the Assent of their Readers; little regarding by what Ambages, by how many circumambulatory Propositions, and almost whole Volumes, it was done; that thereby they might be forced to acknowledge the thing to be so, while how it came to be so, or from what intrinsick Cause or Condition of the Subject requiring it, such and such an Attribute agreed to it, remained in the mean while obscure, or altogether unknown.*

VI.

Hence they made ſuch frequent uſe of Apagogical Demonſtrations, or deductive ad abſurdum & impoſſibile, *which ought not to be done, but where no oſtenſive Demonſtration can be had, or for illuſtrating negative Propoſitions rather than demoſtrating them; for the method of Deduction* ad impoſſibile, *does not ſo much demonſtrate the Truth it ſelf directly, as the conſequent Abſurdity of the oppoſite Suppoſition; whence it follows very indirectly, (tho' moſt certainly) that the Propoſition is true, while in the mean time the original Reaſon of its Truth remains altogether hid and in the dark.*

VII.

But that we may not ſeem unjuſtly to reject the particular Method of the Ancients, made uſe of by Euclid, *as in* lib. 12. Prop. 2, 10. &c. *and by others, but eſpecially by* Archimedes, *who peculiarly addicted himſelf to it, whence it has been by ſome called the* Archimedean *Method, and by* Renaldinus *the Method* per Exploſum exceſſum atque defectum; *beſides its Deduction doubly* ad abſurdum, *whereon it always relies, as* e. g. *it infers the equality of two Magnitudes* A *and* B *by a far fetch round-about way, by ſhewing, that if* B *be ſuppoſed greater or leſs than* A, *from either Poſition there would follow an abſurdity; and thence as it were begging the Equality by a new Inference, which tho' it may paſs free from Suſpicion, yet it neither ought, nor can be admitted, without this Limitation: In comparing thoſe things whoſe Natures are capable of Equality, if it can be demonſtrated that the one is neither greater nor leſs than the other, we may thence juſtly infer their Equality.*

VIII.

The Learned are now generally agreed, that, beſides that Synthetick Method, whereby the Ancients either oſtenſively deduced their Problems and Theorems from evident and common Principles,

ples, or apagogically demonſtrated them by Deductions ad abſurdum, *they alſo made uſe of a certain ſort of Analyſis, whereby they found out thoſe Theorems and Problems; and which, to raiſe the greater Admiration in their Readers, they afterwards ſtudiouſly conceal'd and kept to themſelves: Which Method is undoubtedly preferable to the other, as not only demonſtrating the certainty of the Propoſitions ſo found, but at the ſame time ſhewing the invention of them too; and this is that Method that* Vieta, Harriot, *and* Des Cartes, *and their Followers have not only brought to light in this laſt Age, but to a great degree of Perfection too, and whereof* Carolus Renaldinus, *in that vaſt Work he has intituled* Ars Analytica Mathematum, *has given us a large Treatiſe.*

IX.

There has appear'd moreover of late another particular Method * *invented by* Bonaventura Cavallerius, *which is called* the Method of indiviſibles, *whereby the moſt difficult and abſtruſe Problems of Geometry are found out and Demonſtrated with an incredible eaſe, which is the above-mentioned* Renaldinus's *Opinion of it,* lib. 1. Reſol. & Comp. p. 239. *which, to demonſtrate the Equality or Proportions of Figures and Bodies that may be compared with one another, goes to work after a way which ſeems to be more natural than any other, by ſuppoſing plane Figures to conſiſt of innumerable lines, and ſolids of innumerable Plans (called their indiviſible Parts or Elements, becauſe the Lines are conceived without latitude, and the Plans without any thickneſs,) and relying on this ſelf-evident* Axiom, *That if all the Indiviſibles of one Magnitude collectively taken, be equal or proportional to all the correſpondent Indiviſibles of another, or taken ſeparately each to each, then alſo thoſe Magnitudes will be equal or proportional among themſelves. Which Inference can be guilty of no Fallacy, nor liable to any Error, as long as thoſe Elements are taken and conceived in that ſenſe their Authors deſign them; which is ſufficiently demonſtrated by* Renaldinus, lib. 1. Compoſ. & Reſol. p. 245. *towards the bottom,*

* Dr. *Wallis* is of Opinion this is nothing but an Improvement of the Ancients Method of Exhauſtions.

fom, and at the beginning of p. 306. *and also by* Honoratus Faber, *in his Synopsis,* p. 24. *and Dr.* Barrow *in* Lect. Geom. p. 24. *and the following, against* Tacquet *and the other Adversaries of this method.*

X.

There is another Method a-kin to this, which may be properly named Generative, *very much followed by* Faber *in his Synopsis, and* Barrow *in his* Lect. Geom. *whose Author* Renaldinus *tells us was* Guldinus, lib. Cit. p. 253. *shewing at large its Rules and Foundations in the following pages,* viz. *The rise of Lines from certain motions of points; of Plane and Curvilinear Surfaces, from the determinate progress or rotation of a given Line, and of Solids by the various motions of various Surfaces; the Productions whereof are so represented to the Imagination, that the intrinsick nature of the magnitudes thence arising may become known, and their Properties and Affections may, from their Natures thus known, be easily and briefly deduced.*

XI.

Near a-kin to this Method of Cavellerius *is that other of* Infinite Progressions, *wherein having found a certain Progression of like Parts circumscrib'd about, or inscrib'd in any given magnitude, which may be continued by Bisection* ad infinitum; *and then at length (by vertue of the Doctrine of Exhaustions, founded on* Prop. 1. lib. 10. Eucl.*) will terminate in the magnitude it self, I say, wherein the sum of those infinite Terms, collected by Rules on purpose for the addition of those Progressions, and consequently the quantity or proportion of the proposed Magnitude, to any other given one, may be expressed or defined. But this termination of Figures infinitely circumscribed or inscribed in a Circle, not pleasing* Renaldinus *(altho' his Dissention seems only to consist in Words,) he exhibits another Method like it, (which he peculiarly calls his own) built on twelve fundamental Theorems, and illustrated by several Examples,* lib. 1. de Resol. & Comp. p 277 & seqq.

XII.

Of late also, the most ingenious Mr. Isaac Newton, *to demonstrate his* Philosophiæ Naturalis Principia Mathematica, lib. 1. sect. 1. *premises some Lemma's of his method of* Rationes primæ & ultimæ, *or evanescent quantities, thereby to avoid the tediousness of deducing long and perplext apagogical Demonstrations after the way of the Ancients. For finding that his Demonstrations might be very much contracted by the method of Indivisibles, and knowing at the same time, that that method was scrupled by some, and thought not very Geometrical, he rather chose to found his Method on the sums and proportions of quantities which he calls Evanescent, which performs the same as the Method of Indivisibles, and may be more safely used; which he inculcates in these very words, and others such like, in* Schol. *of* Lemma. 11. *And also answers several objections which might seem to make against it.*

XIII.

But it would be in vain for us to attempt, in this place, to explain all and each of those various methods at length; having only proposed to our selves, to demonstrate the chief and principal Theorems and Inventions of the Mathematicks, and to use sometimes one of them and sometimes another, (having first Demonstrated their Foundations) according as we shall judge this or that of them, fittest to Demonstrate the thing in hand, and so shew the reasons, and use of each of them in the process of this discourse. And altho' H. Faber *in his* Synopsis, p. 8. *Insinuates, that Analytick terms ought not to be made use of in Geometrical Demonstrations, because that Algebraick method seems to be too difficult for young beginners; yet we are of the quite contrary opinion (nay we can scarce doubt but that that Ingenious man would also agree with us herein, if he saw the way we make use of those foundations of Algebra, which is only of the most simple and general principles of it) especially in this case, where the said method is by little and little instill'd with the Demonstrations themselves; and the literal Computations taught from their first*

Princi-

Principles, than which nothing is more easie: And this is that which we design to do, and so use the learner by degrees to this sort of Demonstration, thereby to prepare him the better for the Analytick Geometry of the Moderns, which is the highest apex *of the Mathematicks. But we had rather our Reader should himself find, than we trouble our selves any further to tell him here, how compendiously we Demonstrate the Propositions of Geometry, by the help of these Analytick notes, without the tedious Concatenation of a long Chain of Consequences, which would be otherwise unavoidable.*

XIV.

After this way we design to go through the following Scheme. 1. *We shall deduce several propositions of* Euclid, Archimedes, *and* Apollonius *from our definitions, and the generations of Magnitudes therein proposed; as Corollaries necessarily flowing from them, and confirm'd only by an immediate and simple consequence.* 2. *We shall demonstrate their chief Theorems (for the sake of which they were forced to Demonstrate several others before-hand, the knowledge whereof for their own sake was not so necessary or valuable) without any long series of antecedent Propositions, or Foreign principles, from a few direct and intrinsick Principles of their own. Whence* 3. *It will follow, that after this Method we shall propose things, and treat of them, in a more natural Order, and first of all deliver those which are most universal and common to all quantities, and then descend to those which in a more special manner regard Magnitude, and distribute and dispose all according to certain general distinct Classes of the things to be treated of, and their affections. Hence also* 4. *We deduce from those universal Theorems, by way of Corollary, the Precepts of vulgar Arithmetick, and specious Computation, which afterwards we make use of in particular Demonstrations after a very short and compendious way; and, for this very reason, some Learned men of the present Age are of opinion, that the Ancients often fell into that tedious and intricate prolixity in their Demonstrations, because they would not acknowledge the great affinity there was between Arithmetick*

tick and Geometry, taking particular care not to introduce the Terms and Operations of Arithmetick into Geometry; tho' at the ſame time they never ſcrupled to transfer the names of Plan, Square, Cube, and ſuch like to numbers. 5. *Laſtly, Having firſt Demonſtrated the firſt and Fundamental Theorems of Elementary Geometry, we may ſafely build on them the* Praxes *of all kinds of Mathematical Arts, that are moſt uſeful and requiſite to ſeveral Exigencies of human Life, as, firſt,* Trigonometry *both Plain and Spherical, the Conſtruction and uſe of the Tables of Sines and Tangents.* 2. *The* Conſtruction *of* Logarithms, *and a compendious application of them to* Trigonometry: *And in the* 3 *and laſt place, the fundamental Precepts of Algebra, or the Analytic Art; by the help whereof the learner may at length arrive to the higher and more recluſe parts of Geometry, and become maſter thereof: Not to mention ſeveral Geometrical and Arithmetical Problems, which we have all along derived from ſeveral of our Theorems, by way of Corollary, which it may be ſome other time, may make an Appendix of this work.*

XV.

And thus when we ſhall have Demonſtrated not only the chief Theorems of the Ancient Mathematicians, omitting the unneceſſary crowd of thoſe that are only Subſidiary, but alſo have demonſtratively deduc'd the fundamental Precepts of the moſt neceſſary and uſeful Arts that flow from them, and that are abſtracted from matter, as of Arithmetick, Trigonometry, and Algebra; I hope none will doubt but that in this little Volum we have exhibited, as it were the Nucleus *or Kernel of the pure and genuine Mathematicks (for thoſe other Sciences and Arts which go by the name of mixt Mathematick, are moſt of them parts of natural Philoſophy, from the application of Mathematicks to the Phænomena of nature) and ſo may juſtly bear the name of* Matheſis Enucleata.

XVI. Nor

XVI.

Nor are we ignorant, nor shall we conceal what several Learned men have both proposed and already done, for removing those difficulties and blemishes of the Ancient Mathematical Methods we have just now mentioned. The late Admonishments of the anonymous Authour of L'Art de Penser, *no less ingeniously than modestly delivered,* Part. 4. Chap 9. 10. *of the said Treatise, are sufficiently known; as also the laudable endeavours of* A. Tacquet *and* Honoratus Faber *and several others above mentioned for contracting, new ordering, and more easily and directly Demonstrating the chief Geometrical Inventions of the Ancients. There are moreover extant of a certain anonymous Author,* Elementa Geometrica novo ordine ac methodo feré Demonstrata, *Printed at* London *about* 26. *Years ago. There are also* F. Ignatius Gaston Pardies Elemens de La Geometrie, *&c. Translated into Latin after the third Edition, by the Famous* Schmidtius *Professor at* Geneva: *As also of* F. Mich. Mourgues's, *of the Society of* J. Nouveaux Elemens de Geometrie, abreges par des Methodes particulieres en moins de Cinquante propositions, *&c. There are also several other Essays of reducing the Mathematicks into a better Order and Method, the titles whereof we have only as yet seen; and even while these papers were in the Press, there happen'd into our hands a Treatise of* F. Lamy's *Entituled* Les Elemens de Geometrie, ou de la mesure des Corps, *&c. Printed at* Paris *in* 16‸5: *so that we may only seem to some to do what has been done already, in endeavouring to shew our Reader a new and shorter way to the Mathematicks.*

XVII.

But as none can blame Jacobus le Maire, *because, after the happy discovery of the* Magellanick *Passage from the* Atlantick *into the* Pacifick *Sea, he would needs yet endeavour to find another shorter, which he accordingly did; nor can they be blamed, who now a days consult about finding one from these parts of the*

the World, by the North to the East India's. *Thus also, in an affair of that moment, that one or a few are not sufficient to bring it to Perfection, if any one who comes after, not only invited, but also assisted by the ingenious Endeavours of those who have gone before him, shall undertake to add to their Inventions, to help on the business by his advices, and shew what things are capable of a further Polish, and the method how to perform it, doubtless such an one ought not to be blamed, nor accused of arrogance, unless at the same time he endeavours to depretiate the essays of others, and cry up his own as the only valuable; Which how far it is from our design, the work it self will abundantly shew. Moreover as the senses of men are differently affected by different Objects, and their Palates have different Relishes of the same thing, according to different Preparations of them: So the same truth takes and insinuates it self more easily with one proposed and demonstrated after this way, more with another after that way; and we are so much the more likely to suit the different genii of different Persons, by how many more and different methods and ways we shew them, leading to the same end, of which every one may take that which he likes best.*

XVIII.

We therefore Publish, by the Divine assistance, these our Endeavours also, after so many other ingenious and elaborate ones in the same kind; nor can we doubt the approbation of some of our Readers. This at least we can experimentally affirm: That not a few of those to whom these our thoughts were partly publickly read in Lectures, and partly privately taught (for they were only design'd for Learners) were not a little taken with the Concise *brevity and facility of the* Demonstrations; *so that we may reasonably hope to be acceptable to those, to whom either time, or sufficient force of genius is wanting, to run over the vast Volumes of the Ancient Mathematicians, and comprehend their prolix Demonstrations, and long series's of far fetch'd Consequences; and as for those who have both leisure and genius to do so, this may serve for an Encouragement towards it; that after they have gone through the chief truths and propo-*

sitions

ſitions they contain, Demonſtrated in a more eaſie and ſhorter way, they may ſo much the more confidently adventure upon thoſe celebrated and ingenious Treatiſes, from the reading whereof they were before deterr'd by the length and almoſt inſuperable difficulty of their tedious and perplext Demonſtrations.

XIX.

Being now about to enter upon the matter it ſelf, we will only further hint theſe few things. 1. *Since our whole deſign is for the advantage of young Students (which ought to be a Profeſſors chief Care and Study) we muſt not omit the Explication of the moſt ſimple terms; eſpecially ſince we deſign to deduce ſome Corollaries immediately from them, which heretofore have unneceſſarily increas'd the number of Propoſitions and Demonſtrations.* 2. *To encumber our work as little as we can with words we have made uſe, eſpecially in our Analytick Calculus, of ſome Symbols, as* $=$ *for Equality, as alſo of* □ *and* ▭ *for Square and Rectangle, and of* $\surd$ *the common Radicall ſign for the ſquare Root, with the Line* $\sqrt{}$ *on the top for connecting of quantities together, the Root whereof is jointly taken,* $\sqrt[3]{}$ *for the Cube Root,* $\sqrt{\sqrt{}}$ *for the Biquadratick Root.*

XX.

That the Reader may at one view ſee the Contents of the following Treatiſe, we have thought fit to preſent him here with it, by way of Synopſis. It is divided in two Books.

I.

The firſt whereof contains the chief and moſt ſelect Propoſitions of Euclid's *Elements, of* Archimedes's *Treatiſes of the Sphere and Cylinder, as alſo of the dimenſion of the Circle,* &c. *Wherein that which theſe Authors have demonſtrated by a long and tedious Series of Conſequences, and for the moſt part indirectly, we have here endeavoured to Demonſtrate directly, and ſo that the Demonſtration of each Propoſition depends, either*

either on no other, and ſo is evident by its own light, or on a very few of the antecedent ones.

II.

After the ſame way in the ſecond Book we treat of the Conick Sections, *and Demonſtrate the chief properties of the* Conoid, Spheroid, Cycloid, Conchoid *and* Spiral Lines, *which are extant either in* Apollonius *or* Archimedes *and others, and what they have Demonſtrated by long and tedious proceſs's we have here exhibited in a ſhort and eaſie Compendium. And that,*

III.

In ſuch a method, as does not ſo much require intent and ſevere thinking, as a bare and eaſie inſpection, and application of the Principles of ſpecious Algebra, and method of Indiviſibles. which yet,

IV.

We don't barely ſuppoſe, and remit our Readers to other Books to learn (which would be too troubleſome) but in the Proceſs of the Work it ſelf, they are gradually, and as occaſion preſents, derived frrm their Original Fountain, and firſt Principles.

V

By the ſame way alſo the moſt uſeful and neceſſary Mathematical Praxes *are laid down under the names of* Corallaries *and* Scholia, *the Conſtruction of the Tables of Sines and Tangents taught, the Original and uſe of the Logarithms Demonſtrated, and the Precepts both of Plain and Spherical Trigonometry deduced from their firſt Principles,* &c.

VI.

The Praxis alſo of pure Arithmetick, and of common Decimal, and (which is ſeldom uſed) of Tetractical, as alſo the

Doctrine

Doctrine of Surds, are derived from their first Original. Whereunto,

VII.

As a Complement of the whole Work, we have added an Introduction to the Specious Analysis, or new Geometry of the Moderns, particularly according to the Method of Des Cartes, *but much facilitated by later Inventions, and Comprizing the Precepts of the Art in six or seven Pages, but illustrated with above forty Examples in the different degrees of Equations.*

What the Reader's opinion will be of these our Endeavours, design'd only for the use of young Students, time must teach us. The Author himself at least, amongst his other performances, allows these the first place.

MATHESIS

Mathesis Enucleata:

OR,

The Elements of the Mathematicks.

Book I.

Explaining the First Principles of the Mathematicks; among which are (in the first place) *Definitions*, and some *Consectaries* that flow from them.

CHAP. I.

Containing the Definitions or Explications of the Terms which relate to the Object of Mathematicks.

DEFINITION I.

Mathematicks is the Science or Knowledge of Quantity, and of Beings, as far as they are subject to it, or measurable; and may justly claim the Name of *Universal*, while it is employ'd in Demonstrating those Properties which are common to all or most *Quantities*: But when it descends to the different Species of Quantity, and is busied in contemplating the Affections belonging particularly to this or that Quantity, it is distinguished by various Names, and distributed into various Parts, according to the various diversities of the Objects.

DEFINITION II.

QUantity may be defined in General, whatever is capable of any sort of Estimate or Mensuration, as immediately the Habitudes and Qualities of Things, as *e. g.* the multitude of Stars in the Heaven, or of Souldiers in an Army, the length of a Rope, or Way, the weight of a Stone, the swiftness or slowness of Motion, the Price of Commodities, *&c.* but mediately, the very things themselves wherein those Estimable Qualities are inherent. Whence with the ingenious *Weigelius* we may not incongruously reduce them all to these four Kinds or Genders, *viz.* 1. to *Quanta Naturalia*, Natural Quantities, or such as Nature has furnish'd us with, as Matter with its Extension and Parts, the Powers and Forces of Natural Bodies, as Gravity, Motion, Place, Light, Opacity, Perspicuity, Heat, Cold, *&c.* 2. to Moral *Quanta* or Quantities, depending for the most part on the Manners of Men, and arbitrarious Determinations of the Will; as for Example, the Values and Price of Things, the Dignity and Power of Persons, the Good or Evil of Actions, Merits and Demerits, Rewards and Punishments, *&c.* 3. to *Quanta Notionalia*, arising from the Notions and Operations of the Understanding, as *e.g.* the amplitude or narrowness of our Conceptions, universality or particularity, *&c.* in Logick; the length or brevity of Syllables, Accent, Tone, *&c.* in Grammar: And lastly, to *Quanta Transcendentia*, Transcendent Quantities, such as are obvious in Moral, Notional, and Natural Beings; as Duration, *i. e.* the Continuation of the Existence of any Being; which in Physicks especially is named *Time*, and may be conceived as a Line, *&c.* To these you may moreover add *Unity*, *Multitude* or *Number*, *Necessity* and *Contingency*.

DEFINITION III.

NUmber (whereon we shall make some special Remarks) if it be taken in the Concrete, is nothing else than an Aggregate or Multitude of any sort of Beings; taken abstractedly, it is, as *Euclid* calls it, μονάδων ποσότης, a *multitude*, or (as they call it) *Quotity of Unities*, on the one hand Number, *i. e.* *many* are opposed to *one*; and in that sense Unity is not a Num-

ber:

ber: On the other hand Unity may be esteem'd a Number, since it is no less (if I may be allow'd that term) some *Quotity* than two or three. But as we denote or signifie particular Things, when we speak of them Universally, by the Letters of the Alphabet, A, B, C, (a, b, c,) *&c.* as universal Signs or Symbols of them; so for distinctly and compendiously Expressing the innumerable Variety of Numbers, Men have found out various Notes, the most natural whereof, are Points disposed in particular extended Orders, as . . . to denote Three,

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to denote Nine, *&c.* But that way which is most commodious for Practise, is by the common Notation, or Cyphers, 1, 2, 3, 4, 5, 6, 7, 8, 9. the invention whereof, as we have it by Vulgar Tradition, is owing to the *Arabians.* By a very few of these we express any number tho never so great, by a wonderful, tho now adays familiar, Artifice; the first Inventor of them having Establish'd this as an arbitrary Law, that the first of them shall signifie *unity* or *one*; the 2d *two*, &c. as often as they stand alone; but placed in a row with others, or on the left hand of one or more 0, or noughts, (which of themselves stand for nothing, but fill up empty places) if in the second, before a nought, they denote Tens; if in the 3d. Hundreds; in the 4th Thousands, in the 5th Myriads or Tens of Thousands; in the 7th so many Thousands of Thousands, or Millions; in the 8th Tens of Millions, *&c.* and so onwards, increasing always in decuple Proportion, by Tens, Hundreds, Thousands, *&c.*

COROLLARY I.

HEnce you have a way of expressing or writing any Sum by these Notes, which you may hear expressed in Words; as if we were to express in Notes the year of our Lord, One Thousand Six Hundred Ninety and Nine, it is manifest, that according to the method above described, by placing 9 on the right hand in the first place, and nine again in the second towards the left, six in the third, and one or unity in the fourth, the business will be done. Thus it will be easie to any one with a little attention, to express any Number whatsoever by these Notes; (as suppose that which *Swenterus* proposes, in *Delic.*

Physico-Math. Part. 1. Probl. 75.) Eleven Thousand, Eleven Hundred, and Eleven.

COROLLARY II.

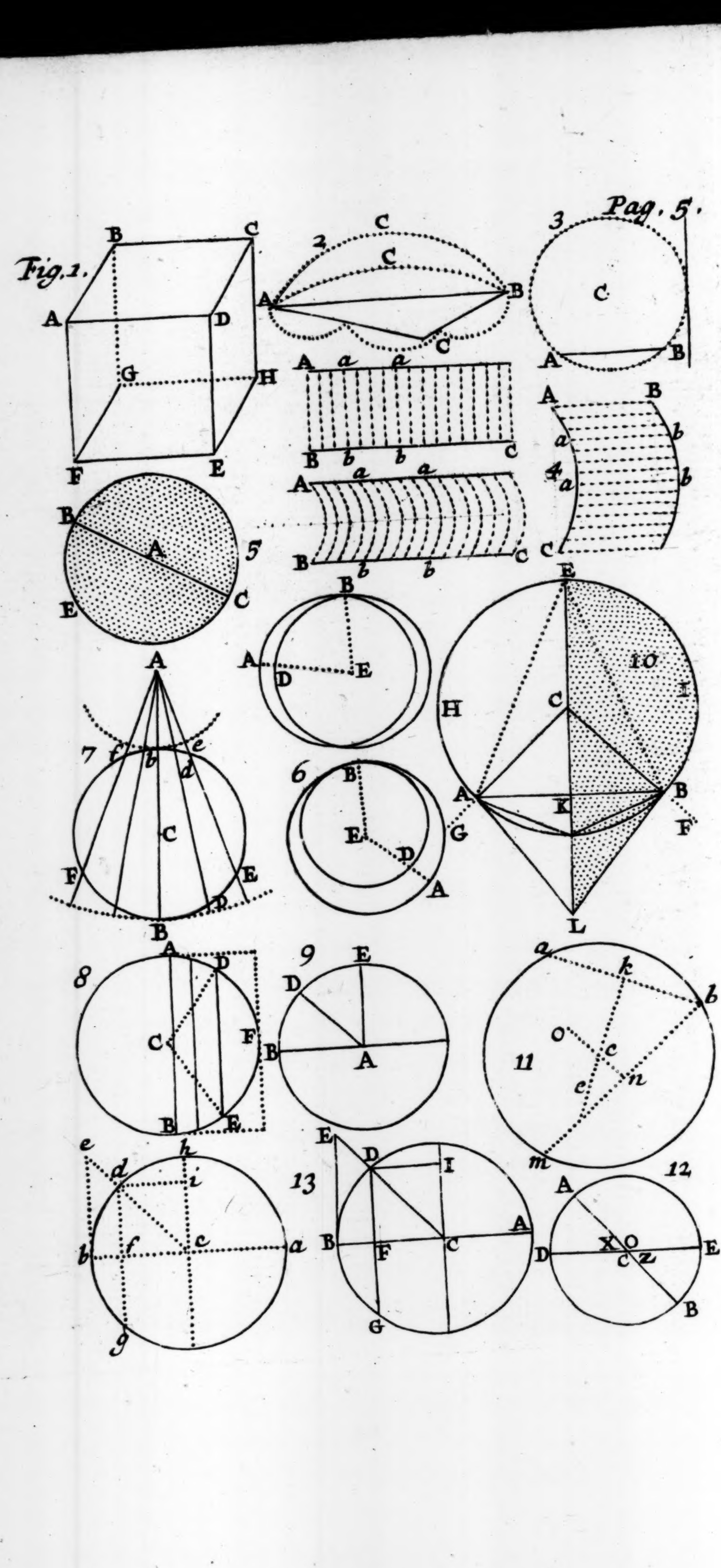
HEnce you have also the Foundation and Reason of the Rule of *Numeration* in Arithmetick, or expressing any Numbers or Sum in Words, which you see written in Cyphers; which for greater ease may be done thus *viz.* beginning from the first Figure towards the right hand, over every fourth Figure note a Point, (including always that which was last pointed) and at every second Punctation or Point, draw short strokes thus ', one over the 2d, two over the fourth, *&c.* the first denoting Millions, the second Millions of Millions, or Bimillions, the third Trimillions, *&c.* and the Intercepted Points the Thousands, in their kinds, *&c.*

SCHOLION.

HEre I cannot omit, on this occasion, what the forementioned *Weigelius* has hinted about another way of *Numeration*, and which Dr. *Wallis* mentions, *Oper Mathemat.* Part 1. p. 25. & 66. shewing there a way (and illustrating it by Examples) of *Numeration*, and of Expressing the Figures; which proceeds thus; whereas now adays in numbring we ascend from unity or 1 to ten (the reason whereof; after which *Aristotle* makes a prolix Inquiry, *Probl.* 3. *Sect.* 15. was taken without doubt from the denary Number of our Fingers) if from unity we proceed only to four, (which *Aristotle* in the same place tells us some of the *Thracians* used to do of old,) and thence returning back again to Unity, we should proceed again after the same way; we might after that way obtain a vastly more simple and easie Arithmetick, than we have now adays. Which, even hence we may conclude; because for Multiplication and Division there would need no other Table (or *Pythagorick Abacus*) than this easie and short one:

1 . 1 . 1 once one is one;
2 . 2 . 10 *i. e.* twice two are four;
2 . 3 . 12 twice three are four and two.
3 . 3 . 21 thrice three are twice four and one

And


Pag. 5.
Fig. 1.

And altho it is pity that we can't hope now a-days to substitute this vastly easier way of Computing, in room of the other now in use, because the other is universally receiv'd, and most sorts of Measures and other Quantities are fitted and accommodated to the decuple Proportion; yet it ought not to be altogether neglected in Mathematicks, which might receive very great advantage hereby, especially in Trigonometry, if the ingenious Invention of the Logarithms had not already supply'd its use therein. The whole Foundation of this *Tetractys* or *Quaternary Arithmetick*, is placed only in these three Notes, 1, 2, 3; so that any one of them alone, or in the first place, shou'd denote Units, in the second place, Tetrads, or so many Fours (or Quaternions,) in the third place so many Sixteens, in the fourth place so many times Four Sixteens, or 64 s. *&c.* always proceeding in a Quadruple Proportion. For which way of Numeration there might be found out terms as commodious as those we now use, and which are thereby grown Familiar to us, as one, ten, twenty, a hundred, a thousand, *&c.* which will be evident by what follows:

One, *Unum* 1 . . . . One.

Ten, *Decem* 10 . . . . . . *Quatuor, Tetras, a Quaternion* or (Four.

Twenty, *Viginti* 20 . . . . . a *Biquaternion*

Thirty, *Triginta* 30 . . . *a Triquaternion*

Hundred, *Centum* 100 . . . *a Tetraquaternion*

Thousand, *Mille* 1000 . . *a Quartan*

Ten Thousand, 10000 *&c.* *a Tetraquartan*, &c.

DEFINITION IV.

A *Magnitude* is whatever is conceived to be Extended or Continuous, or has parts one without another, and contained within some common Term or Terms: wherein that is called a *Point* which is conceived (as indivisible, or) to have no Parts, and so no Magnitude, but is notwithstanding the beginning or first Principle of all Magnitude.

DEFINITION V.

IF we conceive a Point (A) (*Fig.* 1.) to be moved towards B, by this motion it will leave a trace, or describe the Magni-

tude AB of one only Dimension, that is Length without Latitude, or which at least we are to conceive so, and is called a *Line*: If that Line AB be conceiv'd again so to be moved as that its extreme Points AB shall describe other Lines BC and AD, it will describe by that Motion the Magnitude AB CD, or (to denote it more compendiously by the Diagonal Letters) AC or BD, having both length and breadth, but without any depth or thickness, or at least so to be conceived, and this is called *Superficies* or *Surface*: Lastly, if this Surface AC be conceived so to move, *e. g.* upwards or downwards, that its opposite Points A and C again describe other Lines AF and CH, and consequently each of its Lines other Surfaces, &c. by this Motion there will be formed a Magnitude of three Dimensions, which we call a *Solid* or *Body*, which we will also denote by the two diametrically opposite Letters AH and DG. But as this Motion of the Point, Line, or Surface, may be various, so there will be produced by them various sorts of Lines, Surfaces, and Solids: But these Productions stop here, and proceed no further; for the Motion of a Body can only produce another Body greater than the first, but no more new Dimensions.

CONSECTARYS.

I. POints therefore being moved thro' equal Intervals in the same or a like way or trace (*e. g.* in a streight or the shortest trace) describe equal Lines; and

II. The same or equal Lines moved thro' the same Right-lined or Curvilinear Paths, describe equal Surfaces; and

III. Equal Surfaces moved according to the same Methods and Conditions describe equal Solids: which, if rightly understood, are the first certain and infallible Foundations of the Method of Indivisibles. But here you must take care to distinguish between the way which the Line it self describes, and that which its Ends or extreme Points describe: For altho *e. g.* the Point *a* (*Fig.* 24.) moves along in a more oblique way than A, and so describes a longer Line *a c*; yet the Line *a b* describes by a parallel Motion, an equal space with the Line AB, (*viz.*) the same which the whole Line A b, whereof they are parts, would describe. See *Faber's Synopsis*, p. m. 13.

DEFI-

DEFINITION VI.

BUT that we may a little further prosecute this Genesis of Magnitudes (as very much conducing to understand their Nature and Properties) if the Point A moves to B the shortest way, it describes the Right Line AB; but if in any (one) other it will describe the Curve or Compounded Line ACB: From whence, with *F. Morgues*, we may infer these

CONSECTARYS.

I. THat two Right Lines (*a*) beginning from the same Point A, and ending in the same Point B, will necessarily coincide, nor can they comprehend or inclose Space; for if they did, one must deviate, and so would cease to be a Right Line.

(*a*) *Eucl. Ax.* 14.

II. In a Space comprehended by three Right Lines AB, BC, CA, (*a*) any two taken together, must needs be greater than any one alone. Moreover we may add this before hand;

III. In a Circle a Right Line drawn from A to B (*Fig.* 3.) will fall within the Circle, because the Curve Line ADB described, as we shall hereafter shew, being longer than a Right Line, must necessarily fall beyond it, or on the outside of it. And lastly,

IV. A Tangent, or Line (*b*) which does not cut or enter into the Circle, touches it only in one Point.

Moreover if a Right Line AB (*Fig.* 4.) move on another Right Line BC, remaining in the same Position to it, it will generate a *Plan Surface*, to which a Right Line being any way applied, will touch it with all its point, as *Faber* rightly describes it; if a Right Line be moved on a Curve, or a Curve on a Right Line, *&c.* they will generate a *Curve Surface*, call'd *Gibbous*, or *Convex* without, and *Concave* within.

DEFINITION VII.

IF a Right Line be fixed at one of its ends A, and the other be moved round (*Fig.* 5.) it will describe in this Motion a

(a) Eucl. l. 1. Prop. 20.
(b) Eucl. Prop. 2. lib. 3.

Circular Plane, or a Circle; and by the motion of its end or extreme Point B, the *Periphery* or *Circumference* of that Circle BEF. The fixed Point A is called the Center of that Circle; the Lines AB, AC, &c. its *Radii* or *Semi-Diameters*; all of which are equal one to another. Any Right Line BC drawn from one part of the Circumference thro' the Center to another, is called the *Diameter*, and divides the Circle into two Semicircles BECB and BFCB. The Circumference of a Circle, whether great or ſmall, is divided into 360 equal parts called Degrees, and each Degree into 60 Minutes, &c. From this Geniture of the Circle preſuppoſed, there evidently follow theſe

CONSECTARYS.

I. THat 2 Circles which cut one another cannot have the ſame common Center; for if they had, the *Radii* ED and EA drawn from the common Center E (*Fig.* 6.) would be equal to the common *Radius* EB that is the part to the whole.

II. Nor can two Circles touching one another within ſide, have one and the ſame Center, for the ſame reaſon.

(a) Eucl. 5 l. 3
(b) & 6 of the ſame Book.
(c) Eucl. 8 l. 3.

III. Of Lines falling from any given Point without the (*Fig.* 7.) Circle, and (c) paſſing thro' the Periphery to the oppoſite Concave part of it, that which paſſes through the Center of it, is the longeſt, *viz.* AB; and of the other that which is neareſt to it is longer than that which is more remote: But on the contrary of thoſe which fall on the Convex Periphery, that which tends towards the Center, as Ab, is the leaſt, and the reſt gradually greater, and there can be but two, as AE and AF, or Ae and Af, equal: All which will appear very evident by drawing other Circles from the Center A thro' B,D,E, and *b, d, e*. Or thus; having drawn two other Circles, from the Radii AB and A b, if we conceive the Radii Ab and Cb to move towards the right hand, their ends will always recede further from one another; the ſame is alſo evident of the Radii AB and CB, moved alſo to the right together.

IV. Moreover (*Fig.* 8.) of all the Lines drawn within the Circle (a) the Diameter is the greateſt, and the reſt gradually leſs, by how much the more remote they are from the Center, &c.

Which

Which will be very evident to any one who contemplates à Circle inſcribed in a Square, as alſo the Geneſis of Curvity it ſelf; as alſo many other ways which I ſhall now omit; or to mention one more thus; becauſe the two Radii CA and CB being moved, in order to meet together, neceſſarily approach nearer to one another in their extreme Points.

DEFINITION VIII.

THE Aperture or opening of two Lines (*Fig.* 9.) AB, AD, *&c.* that are both fixed at one end at A, and the other ends opened or removed farther and farther from one another, is called an *Angle*, and uſually denoted by 3 Letters, D, A, B, (whereof that which denotes the Angular Point, always ſtands in the middle,) and meaſured by the Arch of a Circle BD, or a certain number of Degrees which it intercepts. The greateſt Aperture of all BAC is when the 2 Legs of the Angle AB and AC make one Right Line, and is meaſured by a Semicircle, or 180 Degrees. The mean or middle Aperture BAE or CAE, when one Leg EA is erected on the other AB or AC at Right Angles, ſo that it inclines neither one way nor the other, (thence called a *Perpendicular*) is named a *Right Angle*, whoſe meaſure is conſequently a Quadrant (or quarter part) of a Circle or 90 Gr. Wherefore a Semicircle is the meaſure of two Right Angles: An Aperture or Angle BAD leſs than a Right Angle (and ſo meaſured by leſs than 90 degrees) is called an *Acute Angle*; and that which is greater than a Right Angle, as DAC (and ſo conſiſting of more than 90 degrees) is called an *Obtuſe Angle.* Whence me may now draw theſe

(*a*) *Prop.* 15. *lib.* 3.

COROLLARYS.

I. TWO or more Contiguous Angles (*a*) conſtituted on the ſame Right Line BC, and at the ſame Point A (as DAB, and DAC or DAB, DAE and EAC) make two Right Angles, as filling the Semicircle; and conſequently,

(*a*) *Eucl.* 13. *lib.* 1. *with the Coroll.*

II. All the Angles that can be conſtituted about the Point A (as filling the whole Circle) are equal to 4 Right ones: As alſo

alſo on the other ſide (*a*) if two Right Lines AB and AD meet on the ſame point A of another Right Line AC, and make the Contiguous Angles equal to 2 Right ones, that is, if they fill a Semicircle, BC will neceſſarily be the Diameter of a Circle, and conſequently a Right line.

III. If one of the Contiguous Angles BAE be a Right one, the other CAE will be ſo alſo.

IV. If two Right Lines AB, CD, cut one another in E, the 4 Angles they make will be equal to 4 Right ones.

V. And as it is evident at firſt ſight (*Fig.* 10.) that any Circle having one half (or Semicircle) folded on the other, at the Diameter ECD, the two Semicircles EHD, and EID, muſt needs agree, or every where coincide one with the other; ſo if the Angle ACD be ſuppoſed equal to the Angle BCD, that is, the Arch AD to the Arch BD, having one Leg CK or CL common; the others AC and BC being ſuppoſed before equal,

1. The Baſes BL and AL, KB and KA, will be alſo equal; for theſe will coincide too, and therefore the Angles alſo.

2. The Line AB being biſected in K, the two Angles (*c*) at K will alſo coincide and be equal, and conſequently Right Angles: and contrarywiſe,

3. The Angles at the Baſe of equal (*d*) Legs, CAB, CBA, and alſo thoſe below the Legs, the Legs being produced to F and G, are equal.

4. Conſequently the Spaces ACL and BCL, ACK and BCK are equal to one another.

(*a*) *Euclid. l.* 1. *Prop.* 14.
(*b*) *Eucl. l.* 1. 4 & 3.
(*c*) *lib.* III. 3.
(*d*) *lib.* 1. 5.

5. The Contiguous Angles AED and BED inſiſting on equal Arches AD and BD are equal, and *è contra*; as alſo thoſe that are not Contiguous, if their Vertex's are equidiſtant from E, &c.

6. It is hence alſo manifeſt, that a Perpendicular erected on the middle of any Line AB, inſcribed in any Circle, paſſes through its Center, by what we have juſt now ſaid; and if you likewiſe erect Perpendiculars on the middle of any 2 Lines, *ab* and *bm* (*Fig.* 11.) connecting any 2 Arches, or any 3 Points, *a*, *b*, *m*, that are not placed all in the ſame Right Line, thoſe 2 Perpendiculars *ke*, *no*, will determine (by their Interſection) the Center of a Circle that ſhall paſs through theſe 3 Points.

DEFI-

DEFINITION IX.

IF one Right Line DE cut or pass thro' another AB (*Fig.* 12) the opposite Angles at the top or intersection ACD and ECB are called *Vertical*; as also the other two ACE and DCB: Whence follow these

COROLLARYS.

I. THat the Vertical Angles are always (*a*) Equal; for both ACD and ECB with the third, ACE, which is common to both, fill or are equal to a Semicircle; as likewise both ACE and DCB with the third ECB, which is common.

(*a*) *Eucl. lib.* 1. 15.

II. Contrarywise, if at (*a*) the Point C of the Right Line DE, the 2 opposite Lines AC and CB make the Vertical Angles x and z equal, then will AC and CB make one Right Line; for, since x and o make a Semicircle, and z and x are equal, by Hypoth. o and z will also make or fill a Semicircle, whose Diameter will be ACB.

III. By the same Argument it will appear, that of 4 Lines (*b*) proceeding from the same Point so as to make the opposite Vertical Angles equal, the 2 opposite ones AC and CB, as also DC and CE, will make each but one Right Line; for since all the 4 Angles together make a whole Circle, or 4 Right Angles, and the sum of x and o is equal (by Hypoth.) to the sum of o and z, it follows, that both the one and the other will make Semicircles, whose Diameter will be AB and DE, and so Right Lines.

(*a*) *The same Prop.* Schol. 1.
(*b*) *Schol.* 2.

DEFINITION X.

IN any Circle, a Right Line, as DG, that subtends any Arch of it DGB, is called the *Chord* of that Arch (*Fig.* 13.) BF (a part cut off from the Semidiameter BC passing thro' the middle of the Chord) is called the *Sagitta* or *Intercepted Ax*, but most commonly the *Versed Sine*; and DF let fall from the other extremity of the given Arch BD, on the Semidiameter at Right Angles, is called the *Right Sine* of that Arch BD, or of the Angle BCD;

BCD; alſo DI is called the Right Sine of the Complement (or for brevity ſake, Sine Compl.) of that Arch DH, or Angle DCH, *&c.* but the greateſt of all the Right Sines HC let fall from the other extremity (or end) of the Quadrant (which is indeed the ſame as the Semidiameter of the Circle) is called the *whole Sine* or *Radius*; laſtly, BE is called the *Tangent* of the Arch BD or Angle BCD, and CE its *Secant:* Whence Mathematicians, for the ſake of Trigonometrical Calculations, have divided the whole Sine or Radius of the Circle into 1000, 10000, 100000, 1000 000, 10000 0000, *&c.* parts, thence to make a proportionable Eſtimate of the number of Parts in the Sine, Tangent, or Secant of any Arch, *&c* as may be ſeen in the Tables of Sines, Tangents, and Secants. From theſe Suppoſitions and Explications of the Terms, we ſhall now infer from this Definition the following

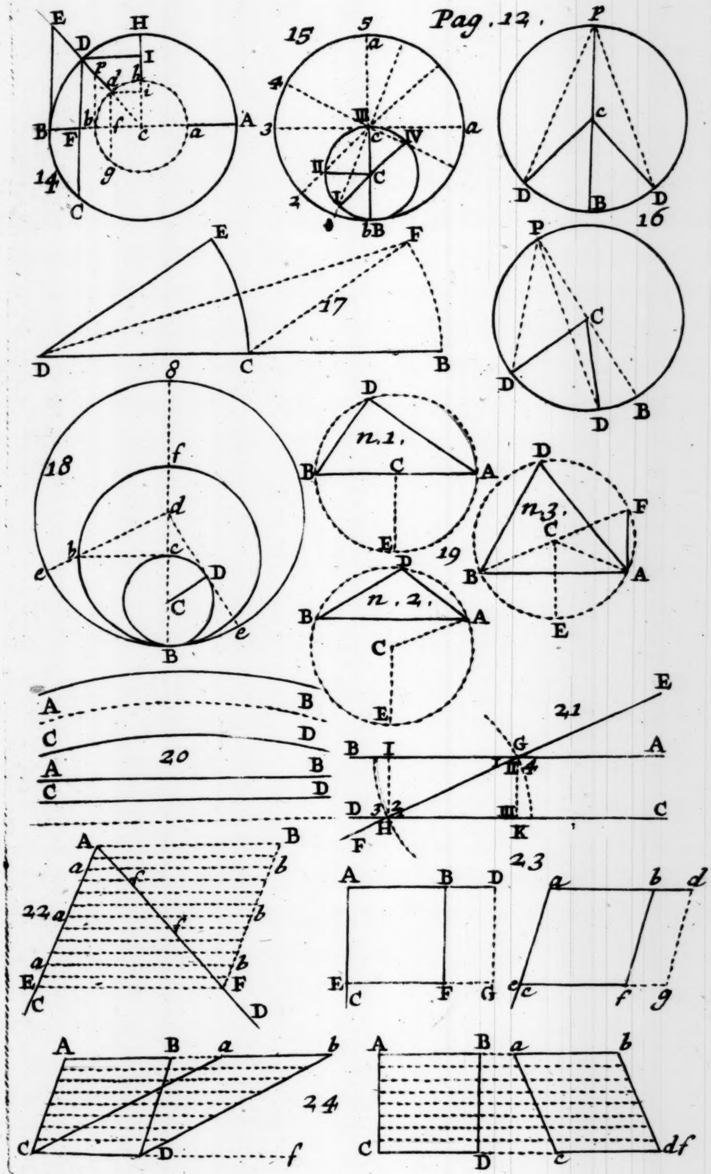
COROLLARYS.

(a) *Among the reſt* Vid. *Eucl.* 28 & 29 *lib.* 3, *and alſo partly the* 26 *and* 27.

I. IN equal Circles (and ſo much more (a) in one and the ſame) as the *Radii* or Semidiameters BC and *b c* are equal, ſo alſo it is evident, that the *Right Sines* DF & *Df*, of equal Arches BD and *b d*, or equal Angles BCD and *bcd*, alſo the Tangents BE and *b e*, and Secants CE and *c e*, and Subtenſes or Chords DG and *d g*, alſo the Sagittæ or intercepted Axes BF and *b f*, of double the Arches DBG and *d b g*, &c. will be equal, and ſo conſiſt of an equal number of Parts of the whole Sine or Radius, *&c.* which both is evident from what we have ſaid before, and may be further evinced, if one Circle be conceived to be put on the other, and the Radius BC on the Radius *b c*, that ſo they may coincide, by reaſon of the equality of the Arches BD and *b d*; and ſo of all the reſt. And *è contra*,

II. In unequal Circles, the Sines, Tangents, *&c.* of equal Angles BCD or *b c d* (*Fig.* 14.) or ſimilar Arches, or Arches of an equal number of Degrees, BD and *b d*, will be alſo ſimilar or like, *&c. i. e.* the Sine *df* conſiſts of as many parts of its *Radius bC*, as the Sine DF does of its Radius BC, *&c. e. g.* if the Radius BC be double of the Radius *b c*, each thouſandth part of the one, will be double of each thou-

ſandth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ſandth part of the other, but they are alike 1000 in each; becauſe the degrees in the Circumference of the little one, particularly in the Arch *b d* are but half as big as thoſe in the Arch BD, and yet equal in number in both. Thus alſo if the Sine DF contains 700 of the 1000 parts of its Radius BC, *df* will alſo contain 700 of the 1000 parts of its ſmaller Radius *bc*, and in like manner the Chords DG and *dg*, and the Tangents BE and *be*, *&c.* contain a like number of parts, each of its own Radius.

SCHOLION.

IT may not be amiſs here to note by the by (altho it may ſeem more proper to be taught after the Doctrin of Proportions) that if, *v. g.* the degrees of a greater Circle be each of them reſpectively double, or triple, or quadruple, *&c.* of the degrees of a leſs Circle, according as the Radius of the one is double or triple to the Radius of the other, then, at leaſt as far as Mechanical Practice can require, you may find the Arch of a greater Circle equal to the whole Periphery of a leſs, *viz.* if you take reciprocally that part of the greater Periphery, which ſhall be as the Radius of the leſs to the Radius of the greater, or as one degree of the leſs Periphery to one degree of the greater. *e.g.* if the leſs Radius *bc* be half the greater BC, and ſo alſo the Periphery, and each of the degrees of the one, be one half of the Periphery, and of each of the degrees of the other, one half of the greater Periphery will reciprocally be equal to the whole leſs Periphery, or 180 degrees of the one to 360 of the other, *&c.*

2. The ſame (at leaſt in this caſe where the Radius *cb* is double of the Radius CB) may be done alſo Geometrically by the ſame reaſon. Having deſcribed Circles on each Radius, ſuppoſe the Radius CB (*Fig.* 15.) ſo to move with an equable motion about its Center *c*, as to take or move the Radius of the greater Circle *cb* along with it, and coming, *v.g.* to I. ſtops that alſo at 1, and going forward, to II. ſtops that again at 2, *&c.* Hence it will be manifeſt to any attentive Reader, that when the leſs Radius CB ſhall have deſcribed the Semicircumference B. II. III. the greater Radius *cb* having moved to 3, will have deſcribed preciſely a quarter of its circumference; and

and if still the less Radius C. IV. moves on to the right hand, and continues to carry the greater *c* 4 along with it onwards the same way, it will necessarily follow, that in the same moment as the Radius C. IV. (together with *c* 4.) shall come to its first situation in B, having described a whole Circle, the opposite Radius *c* 4 will be come to 5, and have described half its Circle, having moved all along with an equal Motion. Hence it is evident, that the whole least Circle answers exactly to half the greater, and half of the first to a quarter of the last; as also the Quadrant B. II. to the Octant (or 8th. part) *b* 2, &c. whence any Arch being given, as B. I. in the least Circle, if you draw thro' I the Radius of the greater Circle *c* 1. you'l cut off an Arch *b* 1. equal to the given Arch in magnitude, but only half in the number of degrees.

3. Hence follows naturally that celebrated Proposition of *Euclid*, that the Angle at the Center BC. I. or BC. II. is double of the corresponding Angle at the Periphery *bc.* 1. or *bc.* 2, *&c.* which in this case is manifest, and in the other 2 (*Fig.* 16.) of the wholes or remainders DCD and DPD it is also (*a*) certain; which is true also of the parts *B*CD and *B*PD to be added or subtracted by the first Case.

4. Hence we have a new way of bisecting any given Angle CDE, or Arch CE (*Fig.* 17.) *viz.* if you make C*B* equal to the Leg DC, and from this, as Radius, describe an Arch *B* F equal to the Arch CE, and draw DF.

(*a*) *Eucl. p.* 20. *l.* 3.
(*b*) *Eucl. p.* 9. *l.* 1.

And with the same facility we might obtain the Trisection, if the greater Radius being triple to the less, was thus carried along by an equable Motion, as we have shewn how to do already in a double Radius; and this at first sight may seem very probable.

*B*ut whether the triple Radius be immediately carried round by the simple Radius C*B*, or by means of the double Radius *c b*, neither the one nor the other will cause an equable Motion. For in the latter Case, while the Radius *c b* describes the quadrant *Bb*, the Radius *d e* will not describe so much as a Quadrant; but while *c b* with the same velocity describes the other Quadrant *b f*, the Radius *d e* will come to *g*, describing an Arch as much greater than a Quadrant as the former

mer was leſs. In the former caſe on the contrary, the Radius C *B* moved on to D beyond a Quadrant, while *de* was carried from *B* to *e*, but if the Radius CD moving on, ſhould again carry *de* along with it, the one would deſcribe the ſame Arch *e B*, while the other would deſcribe one leſs than before.

5. Hence the Angle at the Center ACE (*Fig.* 19.) upon the Arch AE, is equal to the Angle *ADB* at the Periphery, upon double that Arch *AB*.

6. Hence the Angle *ADB* in the Semicircle (Num. 1.) (*a*) is a Right Angle, in a Segment leſs than a Semicircle (Num. 2.) is an obtuſe Angle, and in a greater (Num. 3) an Acute one, becauſe the Angle at the Center *ACE* upon the half Arch, is equal to the Angle *ADB pr. præced.* 5. and is a Right Angle in the firſt Caſe, Obtuſe in the ſecond, and Acute in the third.

(*a*) *Eucl.* 31. *lib.* 2.

7. Hence Angles in the ſame Segment, or (*b*) on equal Segments of equal Circles, or on the ſame or equal Arches, are all equal and *è contra*.

(*b*) *Eucl.* 26 *and* 27. *lib.* 3.

DEFINITION XI.

WHen 2 or more Lines *AB* and *CD* are ſo continued as to keep always the ſame diſtance from one another (whoſe Geneſis may be conceived to proceed from the uniform Motion of 2 Points *A* and *C*, always keeping the ſame diſtance from each other) they are called *Parallels*: But as it evidently follows from this Definition, that (*a*) thoſe Lines which are Parallel to one third, are parallel to one another (ſince adding or ſubtracting equal Intervals to or from other equal ones, the ſums or remainders muſt needs be equal:) ſo if the Parallels are Right Lines and cut tranſverſly (or ſlopingly a-croſs) by another Right Line *E F*, you'l have theſe

(*a*) *Eucl. lib.* 1. 30.

COROLLARYS.

I. THE Angles (*b*) which we call *Alternate* ones, GHK and HGI (*Fig.* 21.) are equal by *Corollary* I. *Definition* X. ſince the diſtances GK and HI, which are the Right Sines of the ſaid Angles, are ſuppoſed equal.

(*b*) *Eucl. lib.* 1. 29.

II. The

II. The External Angle EGA is alſo equal to the Internal oppoſite Angle GHK, by Conſect. 1. *Definit.* 9. becauſe that External Angle EGA is equal to the alternate Vertical one HGI.

III. The ſame Internal Angle GHK, with the other internal oppoſite one on the ſame ſide AGH (as well as the External one EGA, by *Coroll.* 1. *Definit.* 8.) are equal to two Right ones.

IV. On the contrary, If any Right Line EF (*a*) cutting 2 others AB and CD tranſverſly, makes the alternate Angles GHK and HGI equal, their Right Sines, by *Conſect.* 1. *Definit.* 10. will be equal, and conſequently the Lines AB and CD parallel: and the ſame will follow, if the External Angle be ſuppoſed equal to the Internal, or the 2 Internal ones on the ſame ſide equal to 2 Right ones; ſince from either Hypotheſis the former will immediately follow.

(*a*) *lib.* 1. *Prop.* 27 & 28

V. From whence it appears more than one way (*b*) *That the* 3 *Internal Angles of any Triangle* (e. g. H,G,K, which will ſerve for all) *taken together, are equal to two Right ones, and the External one* GHD *is equal to the two Internal oppoſite ones.* For we might either conclude with *Euclid*, that 1, 2, 3, together make 2 Right ones, by *Conſect.* 1. *Definit.* 8. but 2=II and 3=III pr. 1 and 2 of this, therefore I, II, III = 2 Right ones; or with others, 1, II, 4 are =2 R. but 1=I and 4=III pr. 1ſt of this. Therefore, &c. or more briefly with *F. Pardies*, 1=I pr. 1ſt of this, but 1, II, III, together = to 2 Right ones, by the 3d of this; therefore I, II, III = 2R. Q. E D.

(*a*) *lib.* 1. *Prop.* 32,

DEFINITION XII.

IF a Right Line AB (*Fig.* 22.) be conceived to move from the top of a plain Angle CAD with a motion always parallel to its ſelf, ſo that at one end A it ſhall always touch the Leg AC, and all along cut the Leg AD, while at length being come to F, it ſhall only touch that Leg with its other end B, and ſo fall at length wholly within the Angle CAD: It will deſcribe by this motion within the Legs CAD the Triangular Figure EAF, and without them the Triangular Figure BAF; its parts within them *af* continually increaſing, and the other without

fb

fb continually decreasing; but with all its Parts, or the whole Line, it will describe the Quadrangular Figure AEFB: Consequently if the other Leg AD of the given Angle CAD (*Fig.* 23.) or any part of it AB, be moved along the other Leg remaining parallel to it self, it will also describe a Quadrilateral Figure, which will be also equilateral, if the Line describing it AB, be equal to the Line AE according to which it is directed; but if either of the Lines, as AD be greater than the other, the opposite Sides will be only equal; for the describent or describing Line is always necessarily equal to its self, and the Points A, B, D, moved with an equal Motion, describe also in the same time equal Lines AE, BF, DG. From these *Geneses* of Quadrangles and Triangles we have the following

CONSECTARIES.

I. THese Quadrilateral Figures are also Parallelograms, *i. e.* they have their opposite Sides Parallel; (*a*) because the Line that describes them is supposed to remain always parallel to its self, and the Points A and D, or A and B, to be always equidistant.

(*a*) *Schol. Prop.* 34. *lib.* I.

II. Because the 2 Internal opposite Angles (*b*) A and E, and also E and F, &c. are equal to 2 Right ones, by *Consect.* 3. *Definit.* 11. if one Angle *v. g.* that at A be a Right one, all the others must necessarily be so too [in which case the quadrilateral and equilateral Figure AF is called a *Square*, and the other AG an *Oblong*, or *Rectangle*:] if there be no Right Angle, the opposite Angles transversly or cross-ways, *a* and *f*, or *a* and *g* are equal, because both the one and the other, with the third (*e*) make 2 Right ones [in which case the quadrilateral Equilateral *af* is called a *Rhombus*, but the other *ag* a *Rhomboid*.

(*b*) *The first part of the same Proposit.*

III. The Transversal (or Diagonal) Line (*c*) in any Parallelogram, divides it into two equal Triangles AEF and FAB; for all the Lines and Angles on each side are equal, and as the describent (Line) AB moved thro' the Angle EAF upon the Line AE described the Triangle AEF; so the Line EF, equal to the former, moved after the same way, thro' the Angle AFB also equal to the former Angle, upon the equal Line FB, must necessarily

(*c*) *the latter part of the same Prop.*

necessarily describe an equal Triangle; or, in short, all the Indivisibles *a f*, or their whole increasing Series, are necessarily equal to the like number of Indivisibles *f b*, increasing reciprocally after the same way.

IV. All Parallelograms that are between the same Parallels AB and CF (*Fig.* 24.) *i. e.* having the same Altitude (*a*) and the same or equal Bases, as CD or CD and *c d*, are equal among themselves; for they may be conceived to be described by the equal Lines AB and *a b* equally moved thro' the same or equal Intervals of the Parallel Lines; so that all or each of the Indivisibles or Elements AB will necessarily be equal to all and each of the Indivisibles *a b*; for they all along answer one to the other both in number and magnitude.

(*a*) *lib.* 1. *Prop.* 35 & 36

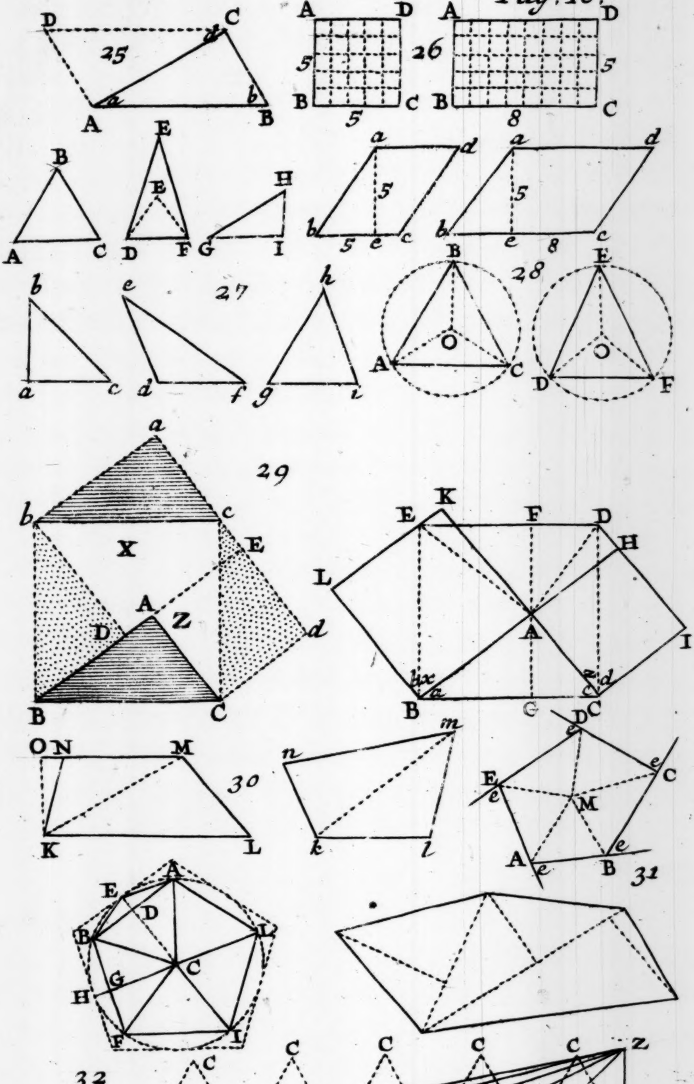
SCHOLIUM.

HEre you have a Specimen of the Method of Indivisibles, introduced first by *Bonaventura Cavallerius*, and since much facilitated; and altho these Indivisibles placed one by another, or as it were laid upon an heap, cannot compose any Magnitude, yet by an imaginary Motion they may measure it, and as it were, after a negative way, demonstrate the Equality of two Magnitudes compared together, *viz.* if we conceive a certain number of such Elements in any given Magnitude, and thence conclude that in another consisting of the like Elements, ordered or ranked after the same way, there can be neither more nor less in number than in the first; thence follows their Equality, &c.

V. Hence therefore it is also Evident, that Triangles upon the same and equal Bases as CD and *cd*, and placed between the same Parallels, are necessarily equal, because they are the half of equal Parallelograms AD and *a d*, by the 3d *Consectary* of this *Definition.*

VI. *F. Mourgues* ingeniously concludes from hence, *viz.* because the 2 Internal opposite Angles (*b*) on the same side in any Parallelogram, are equal to two Right ones, and so all together equal to four; that therefore the three Angles of any Triangle ABC (*Fig.* 25. which may always be compleated into a Parallelogram)

(*a*) *lib.* 1. *Prop.* 37 & 38, & 41.
(*b*) *lib.* 1. *Prop.* 32. *otherwise.*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Parallelogram) are equal to half of thoſe four, or to two Right ones. This may be yet more briefly conceived thus; the Angles $b + c + d$ (the ſum of the two inferiour ones) = to two Right ones; but a = alternate d: therefore $b + c + a$ = to two Right ones, Q. E. D.

VII. Becauſe it is manifeſt in Rectangular Parallelograms, if the Altitude AB, and Baſe BC (*Fig.* 26.) meaſured and divided by the ſame common Meaſure, be conceived to be (multiplied or) drawn one croſs the other, that the * Area AC, thereby deſcribed, will be divided into as many little ſquare Meaſures or Area's, as the number of their Sides multiplied together would produce Units; therefore the Area of any other Parallelogram will be after the like manner produced, if the Baſe be multiplied by the Perpendicular heigth, equally as if it were a Rectangle of the ſame Baſe and Altitude.

* *This is manifeſt from its Geneſis; for the five parts of the Line AB by its motion along the part of the Baſe BE, deſcribes 5 little Squares, and moving along the following part, deſcribes five more, &c.*

VIII. Conſequently alſo you may have the Area of any Triangle, by *Conſectary* 3 and 5. if the Baſe be multiplied by half the Perpendicular heigth; or, the whole Baſe being multiplied by the heighth, if you take the half of the product.

DEFINITION XIII.

BUT as there are various Species of Triangles, while firſt with relation to their Sides, one is called *Equilateral*, as ABC (*Fig.* 27.) becauſe all its Sides are equal; another *Equicrural* or *Iſoſceles*, as DEF, becauſe it has two equal Sides DE and EF, while its Baſe DF may be either longer or ſhorter; and a third is called *Scalenum*, as GHI, becauſe it has all its Sides unequal; then again in reſpect to their Angles, one is called *Rectangled*, as *a*, *b*, *c*, becauſe it has one Right Angle at *a*;

Another *Obtuſangled*, as *d*, *e*, *f*, becauſe it has one obtuſe Angle at *d*; a third is called *Acuteangled*, as *g*, *h*, *i*, becauſe all its three Angles are Acute: So each of theſe kinds has its peculiar properties, which we ſhall partly hereafter demonſtrate in their proper places, and partly deduce here as

CONSECTARYS.

I. ALL Equilateral Triangles are alſo Equiangular, and conſequently Acutangled; for, having found a Center for three Points, and the Periphery A, B, C, (ſee *Fig.* 28.) by *Conſect.* 6. *Definit.* 8. the three Arches AB, BC, and AC, anſwering to equal Chords; and conſequently the three Angles at the Center O are equal, by *Conſect.* 1. of *Definit.* 10. and therefore the three Angles at the Periphery alſo, as being half of the other, by the 3d *Conſectary* of the ſame *Definition.* Each Angle therefore is one third part of two Right ones, by *Conſect.* 6. *Definit.* 12. two thirds of one Right Angle, *i. e.* 60 degrees, and conſequently Acute.

II. It follows alſo by the ſame Reaſon in an *Iſoſceles* Triangle, that the Angles at the Baſe oppoſed to equal Sides are equal, and (*a*) conſequently Acute; for having circumſcribed a Circle about it, equal Arches will correſpond to the equal Chords DE and EF, and equal Angles at the Center DOE and FOE will correſpond to them, and equal ones at the Periphery DFE, and FDE to theſe again. And it is evident that each of theſe are leſs than a Right Angle *h. e.* an Acute one, becauſe all three are equal to two Right ones. Wherefore if the third is a Right Angle, the other two at the Baſe will neceſſarily be half Right ones.

(*a*) *lib.* 1. *Prop.* 5. *the ſame otherwiſe demonſtrated.*

SCHOLIUM.

WE will here (*a*) ſhew by way of Anticipation, the truth of the *Pythagorick Theorem*, eſteemed worth an *Hecatomb*: Which hereafter we will demonſtrate after other different ways; *viz. In a Right Angled Triangle BAC* (*Fig.* 29.) *the Square of the greateſt Side oppoſite to the Right Angle, is equal to the Squares of the other two Sides taken together.* For having deſcribed the Squares of the other two Sides, AC *d E*, DE *ab* (taking ED = A*B*) and the Square of the greateſt *B*C *cb*, it will be evident, that the parts X and Z are common to each, and that the two other Triangles in the greateſt Square *B*AC and *B*D*b*, are equal to the two Triangles *b a c* and C *d c* which remain

remain in the two Squares of the lesser Sides; and so the whole truth of the Proposition will be evident, while these two things are undoubtedly true: 1. That the Side of the greatest Square *Bb* will necessarily concur with the Extremity of the less *Db*, and the other Side of the greatest Square *C c* with its Extremity *c*, will precisely touch the Continuation of the Sides of the two least Squares *dEa*; as you'l see them both expressed in the Figure. 2. The said two Triangles are every way equal; for the Angles at C with the intermediate one at Z, make two Right ones, therefore they are equal; but the Side *CA* is equal to the Side *c d*, and *CB* to *Cc*, and the Angles at *A* and *d* Right ones. Wherefore if we conceive the Triangle *ABC* to to be turned about *C*, as a Center to the right hand, it will exactly agree with the Triangle *C d c*, and the Point *B* will necessarily fall on the continued Line *d E*, as agreeing with the Line *A B*. Hence it is now evident, that *Ca* = *BD*, and because *b a* is also = *bD*, and the Angles at *a* and *D* Right ones. Where, if we conceive the Triangle *b a c* to be moved about *b* as a Center, untill *b a* coincides with *bD*, and *a c* with *DB*, *b c* will also necessarily coincide with *b B* *Q. E. D.*

To this Demonstration of *Van Schooten*'s, which we have thus illustrated and abbreviated, we will add another of our own, more like *Euclids*, but somewhat easier, which is this: Having drawn the Lines (as the other Figure 29 directs) the △ *ACD* being on the same Base *AC* with the Square *A I*, and between the same Parallels, is necessarily one half of it, but it is also half of the Parallelogram *CF* being on the same Base with it, *viz.* *DC*; therefore this Parallelogram = ▭ *AI*. In like manner △ *ABE* is half the ▭ *A L*, and also half the Parallelogram *BF*, therefore *BF* = ▭ *AL*: therefore *CF* + *BF* that is the ▭ of *BD* = to the two ▭▭ *AI* + *AL*. *Q. E. D.* For because the Side *B E* occurs to, or meets the Side *LK*, and the Side *CD* the Side *IH* continued, it yet more apparently follows; because the Angles *a* and *b*, and also *c* and *d*, are manifestly equal, as making both ways, with the Intermediate *x* or *z*, Right Angles. Therefore the △ *B A C* being turned on the Center *B* and laid on *BLE* will exactly agree with it, and turned on the Center *C* and laid on *CID*, will agree with that also, &c.

DEFINITION XIV.

ALL Rectilinear or Right Lined Figures that have more than three or four Sides (to the latter sort of which, there remains to be added another Species besides Parallelograms, call'd *Trapeziums*, whose Angles and Sides are unequal, as K, L, M, N, *Fig.* 30.) are called by one common Name *Poligons*, or *Many-sided* and *Many-Angled Figures*, and particularly according to the Number of their Sides and Angles, *Pentagons*, *Hexagons*, *Heptagons*, &c. All whereof, as also *Trapezia*, being resolvible into Triangles by Diagonal Lines, (as may be seen in the 31 and foregoing Fig.) you have these

CONSECTARYS.

I. YOU have the Area of any Polygon by resolving it into Triangles, and then adding the Area's of each Triangle found by *Consect.* of *Definition* 12 into one Sum.

II. The Area of the *Trapezium* KLMN (in the first of the *Fig.* 30) whose two opposite Sides, at least KL and MN, are Parallel, may be had more compendiously, if the Sum of the Sides be multiplied by half the common heigth KO.

SCHOLIUM.

HEnce we have the foundation of *Epipedometry* or Masuring of Figures that stand on the same Base, and Ichnography; in the Practise whereof this deserves to be taken special Notice of, that to work so much the more Compendiously, you ought to divide your Figure into Triangles, so that (*Fig.* 31.) 2 of their Perpendiculars may (as conveniently can be) fall on one and the same Base. For thus you'l have but one Base to measure, and 2 Perpendiculars to find the Area of both: But for Ichnography, the distance of the Perpendiculars from the nearest end of the Base must be taken; which we shall supersede in this Place and Discourse more largely on hereafter.

2. This resolution of a Polygon into Triangles may be perform'd by assuming a point any where about the middle, and making the sides of the Polygon the Bases of so many Triangles;

Triangles; (see the 2*d* Figure mark'd 31) wherein it is evident; 1 That all the Angles of any Polygon are equal to twice so many right ones, excepting 4, as the Polygon has sides; for it will be resolv'd into as many Triangles as it has sides, and each of these has its Angles equal to 2 right ones. Subtracting therefore all the Angles about the Point M (which always make 4 right ones by *Conf.* 2. *Def.* 8.) there remain the rest which make the Angles of the Polygon. 2 All the external Angles of any right lined Figure (e, e, e, *&c.*) are always equal to 4 right ones; for any one of them with its Contiguous internal Angle is equal to 2 right ones *pr. Consect.* 1 of the said *Def.* and so altogether equal to twice so many right ones as there are Sides or internal Angles of the Figure. But all the internal Ones make also twice so many right Ones, excepting 4 therefore the external Ones make those 4.

DEFINITIOF XV.

Among all these plain Figures those are call'd *Regular* whose Angles and Sides are all equal, as among trilateral Figures the Equilateral Triangle, among Quadrilateral ones, the Square, and in other kinds, several Species which are not particulariz'd by Names; but all others in whose Angles or Sides there is any inequality, are call'd *Irregular*: Tho' some of these also, and all the other may be inscrib'd in a Circle. Whence you have these

CONSECTARYS.

I. THE Areas of the Regular Figures may be obtained yet easier, if having found their Center (by *Consect.* 6. *Definit.* 8.) you draw from thence the Right Lines CB, CA, *&c.* (*Fig.* 32.) till there be form'd as many Triangles ACB, ACF, *&c.* as the Figure has Sides; for since all these Triangles have their Bases AB, BF, as so many Chords, and their Altitudes CD, CG, as so many parts of intercepted Axes DE and GH, and also equal *pr. Consect.* 1. *Definit.* 10. and so by *Consect.* 5. *Definit.* 10. are equal among themselves; one of their Area's being found and multiplied by the number of Sides, or half the Altitude by the Sum of all the Sides, you'l have the Area of the

whole Polygon: For it is manifest from what we have already said, and very elegantly Demonstrated by *F. Pardies, That any Regular Polygon inscribed in or circumscribed to a Circle, is equal to the Triangle* AZa, *one Legg whereof is equal to the Perpendicular heighth let fall from the Center upon any Side, and the other to the whole Periphery of the Polygon.* Now if the Triangles into which the Polygon is resolved, do all stand on the same Right Line Aa, (*Fig.* 32) and are all equal and of the same heighth, to which the Perpendicular AZ is equal, it will necessarily follow, that each pair of Triangles ABZ and ABC, BZF and BCF, *&c.* are equal among themselves, *pr. Consect.* 5. *Definit.* 12. and consequently the Sum of all the former will be equal to the Sum of all the latter, that is, the Triangle Aza to the Polygon given.

II. Since Regular Figures inscrib'd in a Circle, by bisecting their Arches AB, BF, *&c.* may be easily conceived to be changed into others of double the number of Sides, (as a Pentagon into a Decagon, *&c.*) and that *ad Infinitum*; a Circle may be justly esteemed a Polygon of infinite Sides, or consisting of an infinite Number of equal Triangles, whose common Altitude is the Semidiameter of the Circle: So that the Area of any Circle is equal to a Right Angled Triangle (as AZa) one of whose Sides AZ is equal to its (*a*) Semidiameter, and the other Aa to its whole Circumference.

(*a*) *Archimedes of the Dimension of the Circle*, *Prop.* 1.

SCHOLIUM.

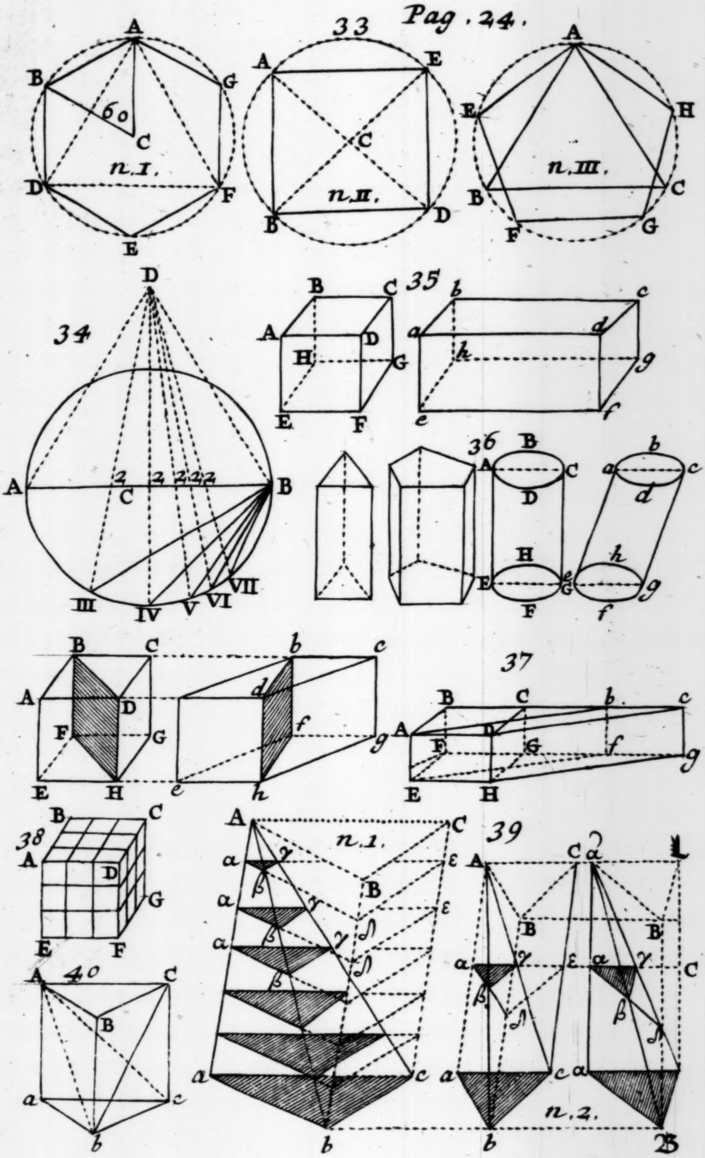
IT may not be amiss to note these few things here, concerning the Inscription of Regular Figures in a Circle.

I. Having described a Circle on any Semidiameter AC, (*a*) (*Fig.* 33. *N.* 1.) that Semidiameter being placed in the Circumference, will precisely cut off one sixth part of it, and so become the Side of a Regular Hexagon: and so the Triangle ABC will be an Equilateral one, and consequently the Angle ACB and the Arch AB 60 Degrees, by *Consf.* 1. *Definit.* 13.

(*a*) *Euclid. Prop.* 15. *lib.* 4.

II. Hence a Right Line AD, omitting one point of the division B, and drawn (*b*) to the next D, gives you the Side of a Regular Triangle inscrib'd in the Circle, and subtends twice 60, *i. e.* 120 Degrees.

(*b*) *Eucl.* 2d *Corol. of the same.*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III. If the Diameters of the Circle AD and DE (*N.* 2.) cut one another at Right Angles in the Center C, the Right Lines AB, BD, *&c.* will be the Sides of an inſcribed Square ABDE: For (*c*) the Sides AB, BD, *&c.* are the equal Chords of Quadrants, or Quadrantal Arches, and the Angles ABD, BDE, *&c.* will be all Right ones, as being Angles in a Semicircle (*per Schol.* 6. *Definit.* 10.) compoſed each of two half Right ones, by *Conſect.* 2. *Definit.* 13.

(*c*) *Eucl.* 6. *lib.* 4.

IV. *Euclid* very ingeniouſly ſhews us how to Inſcribe a Regular Pentagon alſo, *Lib.* 4. *Prop.* 10 & 11. and alſo a Quindecagon (or Polygon of 15 Sides) *Prop.* 16. But though the firſt is too far fetch'd to be ſhewn here, yet (ſuppoſing that) the ſecond will eaſily and briefly follow:

In a given Circle from the ſame point A (*N.* 3.) inſcribe a Regular Pentagon AEFGHA, and alſo a Regular Triangle ABC; then will BF be the Side of the *Quindecagon*, or 15 Sided Figure. For the two Arches AE and EF make together 144 Degrees, and AB 120: (*a*) Therefore the difference BF will be 24, which is the 15th. part of the Circumference.

V. The Invention of *Renaldinus* would be very happy, if it could be rightly Demonſtrated; (as he ſuppoſes it to be in his Book of the Circle) which gives an Univerſal Rule of dividing the Periphery of the Circle into any number of equal Parts required, in his 2d Book *De Reſol. & Comp. Mathem.* p. 367. which in ſhort is this: Upon the Diameter of a given Circle AB *Fig.* 34.) make an Equilateral Triangle ABD, and having divided the Diameter AB into as many equal Parts, as you deſign there ſhall be Sides of the Polygon to be Inſcribed, and omitting two, *e. g.* from B to A, draw thro' the beginning of the third from D, a Right Line, to the oppoſite Concave Circumference, and thence another Right Line to the end of the Diameter B, which the two parts you omitted ſhall touch thus, *e. g.* for the Triangle, having divided AB into three equal parts, if omitting the two B2, thro' this beginning of the 3d you draw the Right Line DIII, and thence the Right Line III.B, which will be the Side of the Triangle; and ſo IV.B will be the Side of the Square, VB the Side of the Pentagon, *&c.*

N. B. The Demonstration of these (*Renaldinus* adds, *p.* 368.) we have several ways prosecuted in our Treatise of the Circle: Some of the most noted Antient Geometricians, have spent a great deal of pains in the Investigation and Effection of this Problem, and several of the Moderns have lost both time and pains therein: Whence, we hope, without the imputation of Vain Glory, we may have somewhat obliged Posterity in this point.

DEFINITION VI.

IF the Plane of any Parallelogram AC (*Fig.* 25.) be conceived to move along a Right Line AE, or another Plane AF downwards, remaining always Parallel to its self; there will be generated after this way a Solid having six opposite Planes Parallel, two whereof, at least, will be equal to one another, whence it is called a *Parallelepiped*; and particularly a *Cube* or *Hexaedrum*, if the *Parallelogram* ABCD that describes it be a Square, and the Line along which it is moved, AE, equal to the Side of that Square, and Perpendicular to the describing Plane, and consequently all the six Parallel Planes comprehending this Solid, equal to one another. But if the describing or Plane Describent (*Fig.* 36.) be a Triangle or Polygon, the Solid is call'd a *Prism*, if a Circle, it is called a *Cylinder.* Now from the Genesis of these Solids you have the following

CONSECTARYS.

I. IF the Planes or Parallelograms Describent (*a*) ABCD and *a b c d* (*Fig.* 37.) are equal, and their Lines of Motion AE and *a e* also equal; the Solids thereby described, *viz.* Parallelepipeds, Cylinders, and Prisms, (which will therefore have their Bases and heigths equal) will be equal among themselves; because the describent Indivisibles of the one, will exactly answer, both in number and position, to those of the other, as we have already shewn in Parallelograms; Consequently therefore,

(*a*) *Eucl. l.* 11. p. 29, 30, 31

II. Any Parallelepiped (*b*) may be divided by a Diagonal Plane BDHF (or a Plane passing thro' its Diagonals) into two equal Prisms; for by

(*b*) *Eucl. l.* 11. p. 28.

Consect

Consect. 3. *Definit.* 12. the Triangles ABD and BCD, are equal, and are ſuppoſed to be moved by an equal Motion thro' equal ſpaces.

III. And ſince it is evident, even by this Geneſis of them, that in Right Angled Cubes and Parallelepipeds, if the Baſe ABCD (*Fig.* 38.) being divided into little ſquare Area's, be multiplied by the heigth AE, divided by a like meaſure for length, after this way you may conceive as many equal little Cubes to be generated in the whole Solid, as is the number of the little Area's of the Baſe multiplied by the number of Diviſions of the ſide AE; you may moreover obtain the Solidity of any other Parallelepipeds, that are not Right Angled ones, by multiplying their Baſes and Perpendicular Heigths together.

IV. Moreover ſince every Triangular Priſm is the half of a Parallelepiped, and any Multangular Priſm may be reſolved into as many Triangular ones, as its Baſe contains Triangles: you may obtain the Solidity (or Solid Contents) either of the one or the other, if you multiply the Triangular, or Multangular Baſe of them into their Perpendicular Heigth.

V. After the ſame manner you may likewiſe have the Solidity of a Cylinder, which may be conſidered as an Infinite Angled Priſm, juſt as the Circle is as an Infinit-Angled Polygon.

DEFINITION XVII.

IF any Triangle ABC (*Fig.* 39. *N.* 1.) be conceived to move with one of its Plane Angles C, from the Vertex or top of a Solid Angle (determined by two Planes aAb and cAa joined together in the common Line Aa) with a motion always parallel to it ſelf; ſo that its extreme Angular Point A ſhall always remain in the Line Aa, but with its Sides AB and AC ſhall all along raze on the two Angular Planes, till at length it falls wholly within the Solid Angle: by this its motion it will deſcribe within the Solid Angle, the Figure we call *Pyramidal*, whoſe Baſe will be the Triangle *a b c*, and its Vertex A will alſo deſcribe without it another Quadrangular Pyramid, whoſe common Vertex will be the ſame A, but the Baſe the Quadrangle Cb, deſcribed by the Side of the moveable Triangle BC: The firſt Pyramid it will deſcribe with its Triangular Parts, $\alpha\beta\gamma$ continually increaſing

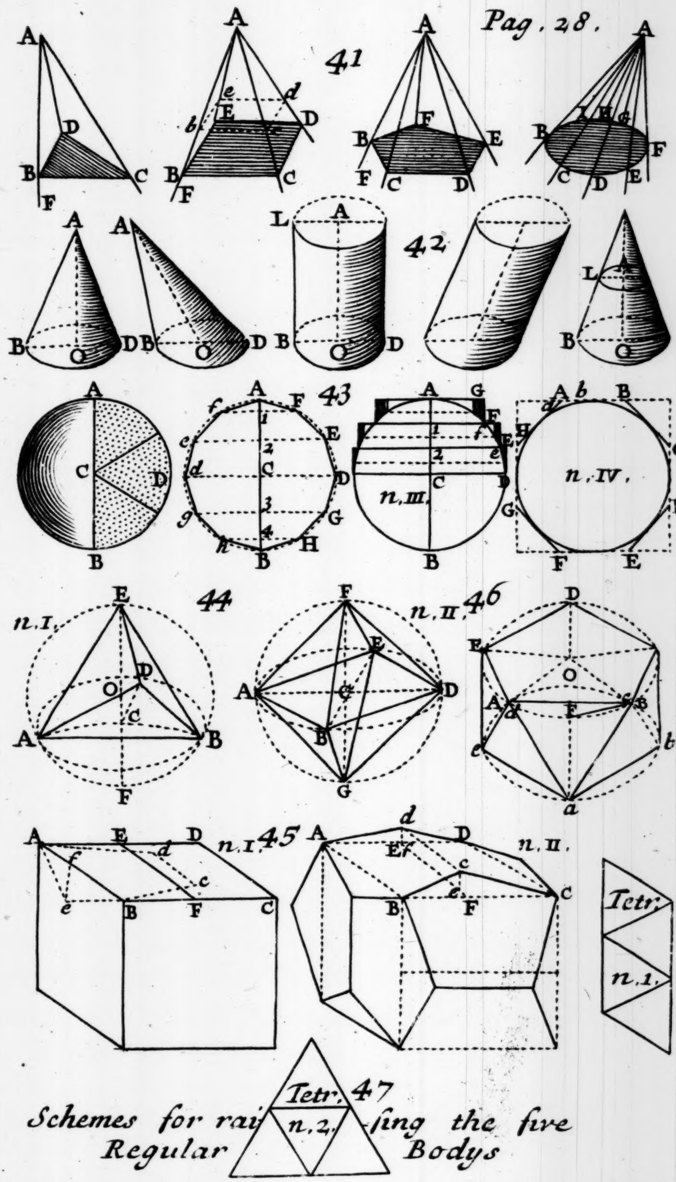
creasing from the point A, and ending in the Triangle *a b c*; but the latter Pyramid will be described by the remaining parts, continually decreasing downwards from the whole ABC, and the Quadrangular Trapeiza *βγδε* at length ending in the Right Line *b c*: So that in the mean while the said Triangle with its whole space describes, according to what we have said before, the Triangular Prism composed of those two Pyramids. From this Geeesis of Pyramids you'l have the following

CONSECTARYS.

I. OF what sort soever the Describing Triangles are ABC, 𝔄𝔅ℭ (N. 2.) so they are equal; and whatever the Solid Angles are, comprehended under the Planes abA, and acA, 𝔞𝔟𝔄 and 𝔞𝔠𝔄, so they are accommodated to the Plane Angles A and 𝔄, and such that, &c.

DEFINITION XVIII.

THere may be exhibited another easier Genesis of Cones and Pyramids,but it respects only the Dimension of the Surface, and not of the Solidity of them, *viz.* If you have a fix'd point A that is not in the Angular Plane BCDEF (*Fig.*41.) and a Rght Line AF let fall from that point to any Angle of the Plane,be conceived to move round the sides BC, CD, &c. This Plane by its motion will describe as many Triangles ABC, CAD, &c. as the Angular Plane has Sides. And these Triangles all meeting at the point A, make that Solid which we call a *Pyramid.* Now if instead of the Angular Plane there be supposed a Circular one, (or an Angular one of Infinite Sides) the Solid thence produc'd is called a *Cone,* whose Surface is equal to Infinite Triangles, constituted on the Base BCDE, and whose Solidity would consequently equal an Infinite Angled Pyramid of the same heigth. And after the same manner by the motion of the Line AF, remaining always Parallel to it self about Parallelograms or Triangular Planes, will be generated Parallelipipeds, Prisms, and Cylinders. But as one Pyramid will be produced more upright than another, according as the point A stands more over the middle of the Plane BCD, &c. (*Fig.* 42.) or respects



Schemes for raising the five Regular Bodys

respects it more obliquely: So in particular a *Cone* is called a *right Cone*, when the line AO being let fall to the Center of the circular Plane (otherwise call'd the Axis of the Cone) constitutes on all sides right Angles with it; but it is call'd an *Oblique* or *Scalene* Cone, when the Ax stands obliquely on the Base. Which distinction may also be easily understood when apply'd to Cylinders, tho' a right Cone and Cylinder may also be conceiv'd to be generated after another way, as if for the one a Triangle and for the other a right angled Parallelogram AOB and LAOB be be conceiv'd to be moved round a line AO considered as immoveable (whence it is call'd an Ax;) and also a truncated Cone may be formed if a right angled Trapezium, 2 of whose Sides are parallel be moved, &c. And as we have deduc'd the Solidity of these Bodies from the foregoing Genesis, so their external Surfaces, as also of Prisms and Parallelepipeds may easily be found from the present Genesis, by any one who attentively considers the following

(a) *Archimedes lib.* 1 *de Con. and Cylin. prop.* 7. *and* 8.

COROLLARYS.

I. SInce the whole external Surface, except the Base, of any Pyramid is nothing but a System of as many Triangles ABC, CAD &c. as the Pyramid has Sides; if the Area's of those Triangles separately found by *Consect.* 8. *Def.* 12. be added into one Sum, you'l have the Superficial Area of the whole Pyramid.

II. If a Pyramid be cut with a Plane *b*, *c*, *d*, *e*, parallel to its Base BCDE (*Fig.* 41.) The Surface of that truncated Pyramid comprehended between the prallel Planes may be obtain'd if having found the Surface of the Pyramid A*b c d e* cut off from the rest by *Consect.* 1. you subtract it from the Surface of the whole Pyramid.

II. The external Surface of a right Pyramid that stands on a regular Polygon Base is equal to a Triangle, whose Altitude is equal to the Altitude of one of the Triangles which compose it, and its Base to the whole Circumference of the Base of the Pyramid.

IV. Therefore the Surface of a right Cone, by what we have already said, is equal to a Triangle whose heighth is the

side

side of the Cone, and the Base equal to the Circumference of the Base of the Cone.

V. The Surface of a truncated right Cone, or Pyramid is equal to a Trapezium which has 2 parallel Sides the lowest of which is equal to the Perifery of the Base, and the other to the Perifery of the Top or upper Part, and the heigth, to the intercepted Part.

VI. The Surface of a right Cylinder or Prism is equal to a Parallelogram which has the same height with them, and for its Base a right line equal to the Perifery of that Cylinder or Prism.

DEFINITION XIX.

IF a semicircular Plane ADB (*Fig.* 43. *N.* 1.) be conceived to move round its Diameter AB which is fixt, as an Axis, by this Motion it will describe a *Sphere*, and with its Semicircumference the Surface of that *Sphere*; every part whereof is equally distant from the middle Point of that Axis C (which is therefore call'd the Center of that *Sphere*.) Now if (*N.* 2.) this semicircular *Ambitus* (*a*) be conceived to be divided before that revolution, first into 2 Quadrants AD, and BD, and then each of those again into as many parts equal in Number and Magnitude as you please, and having drawn the Chords AF, FE, ED, &c. let the Polygon AFEDGHB Inscribed in the Semicircle be conceived together with it to be turn'd about the Axis AB; then will A 1 F, and B 4 F describe 2 Cones about the Diameters F f, and H h; and the Trapezia abont the Axes 1, 2, 2 C, C 3, and 34, will describe so many truncated Cones, and the lines AF, FE, &c. so many Conical Superficies, by the Antecedent *Def.* and so the whole Polygonal Plane AFEDGHB a Conical Body inscribed in the Sphere, and contain'd under only Conical Surfaces. And as any attentive Person may easily perceive such a Body to be less than the ambient Sphere, and its whole Surface less than the Surface of the ambient Sphere; so he may as easily trace these following

(*a*) *Archimedes lib.* 1. *de Cono & Cyl. Prop.* 22. *Coroll. & Prop.* 23.

CON

CONSECTAYS.

I. IF the Arches AF, FE, *&c.* be further bisected, and a Polygonal Figure of double the number of Sides inscribed in the Semicircle, and conceived to be moved round after the way we have already shewn, the Pseudoconical Body hence arising, will approach nearer and nearerto the Solidity of the Sphere, and the Surface of the one to the Surface of the other, and hence (if we continue this Bisection, or conceive it to be continued *ad Infinitum*) you may infer.

II. That a Sphere may be look'd upon as much a Pseudoconical Body, consisting of infinite Sides, and it's Surface will be equal to the infinite Conical Surfaces of that Body; which we will take further notice of below.

DEFINITION XX.

IF the Diameter AB of the Semicircular Plane ADB (*Fig.* 43. N. 3.) be conceived to be divided into equal Parts (as here the Semidiameter AC into 3.) and if the circumscribing Parallelograms CE, 2 E, 1 G on the transverse Parallels CD, 2 *e*, 1 *f* be conceived together with the Semicircle it self to revolve about the fixed Ax AB; it is evident that there will be formed from the Semicircle a Sphere as before, and from the Circumscribed Parallelograms, so many circumscribed Cylinders of equal heighth: but if all the Altitudes or Heighths of these are bisected or divided into two, and so make the number of circumscribing Parallelograms double, there will be formed (by moving them round as before) double the Number of Cylinders of half the heighth, but which yet being taken together, approach much nearer the solidity and roundness of the Sphere, than the former, which were fewer in Number (*viz.* the six latter Parallelograms approach nearer to the Plane of the Circle than the three former) and thus if that bisection of the Altitudes be conceived to be continued *ad infinitum*, the innumerable Number of those infinitely little Cylinders will coincide with the Sphere it self. Moreover if you conceive any Polyedrous or Multilateral Figure to be circumscribed about the Sphere (which we here endeavour to delineate by the Polygon ABCD N. 4. circumscribed about the Cir- cle)

cle) and the solid Angles thereof to be cut by other Planes *ab*, which shall touch the Sphere; it is manifest there will thence arise another Polyedrous Figure, the Solidity whereof will approach nearer to the Solidity of the Sphere, and its Surface to the Spherical Surface than the former, and if the Angles of this be again in like manner cut off, there will still arise another new Solid, and new Surface approaching yet nearer to the Solidity and Surface of the Sphere than the former, *&c.* and so after an infinite Process they will coincide with the Sphere and its Surface themselves. Whence flow these

COROLLARYS.

I. THE Sphere may be considered as a Polyedrous Figure, or as consisting of innumerable Bases, *i. e.* composed of an innumerable Number of Pyramids, all whose Vertex's meet in the Center, and so whose common heigth is the Semidiameter of the Sphere, and the sum of all the Bases equal to the Superficies of the Sphere.

II. If you can find a Proportion between a Cylinder of the same heigth with any Sphere, and whose Base is equal to the greatest Circle of that Sphere, and innumerable Cylinders circumscribed about it, as we have just now shewn; then you may also obtain the Proportion between the said circumscrib'd Cylinder and the inscrib'd Sphere: Which to have here hinted may be of service hereafter in its proper place.

DEFINITION XXI.

THere remain those Bodies to be consider'd which are call'd *Regular*, which correspond to the Regular Plane Figures; and as those consist of equal Lines and Angles, so these likewise are comprehended under Regular and Equal Planes meeting in equal solid Angles; and as those may be Inscribed and Circumscribed about a Circle, so may the latter likewise in and about a Sphere. But whereas there are infinite Species of Regular Plane Figures, there are only five of Regular Solids; the first whereof is contained under four Equal and Equilateral Triangles, whence it is nam'd a *Tetraedrum*; the second is terminated by six equal Squares, and thence is call'd *Hexaedrum*, and other-

wise

wiſe a *Cube*; the third being comprehended under eight Equal and Equilateral Triangles, is call'd an *Octaëdrum*; the fourth is contained under twelve Regular and Equal Pentagons, and ſo is nam'd a *Dodecaëdrum*; the fifth, laſtly, is contained under twenty Regular and Equal Triangles, and is thence nominated an *Icoſaëdrum*. Beſides theſe five ſorts of Regular Bodies there can be no other; for from the concourſe of three Equilateral Triangles ariſes the Solid Angle of a *Tetraëdrum*, from four the Solid Angle of an *Octaëdrum*, from five the Solid Angle of an *Icoſaëdrum*; from the concourſe of four Squares you have the Solid Angle of an *Hexaëdrum*; from that of three Pentagons you have the Solid Angle of a *Dodecaëdrum*; and in all this Collection of Plane Angles, the Sum does not ariſe ſo high as to four Right ones. But four Squares, or three Hexagons meeting in one Point, make preciſely four Right Angles, and ſo by *Conſect.* 2. *Definit* 8. would conſtitute a Plane Surface, and not a Solid Angle. Much leſs therefore could three Heptagons or Octagons, or four Pentagons meet in a Solid Angle, to form a new Regular Body; for thoſe added together would be greater than four Right Angles. But now, for the Meaſures of theſe five Regular Bodies, take the three following

CONSECTARYS.

I. SInce a Tetraëdrum is nothing elſe but a Triangular Pyramid, and an Octaëdrum a double Quadrangular one, their Dimenſion is the ſame as of the Pyramids in *Schol.* of *Definit.* 17.

II. The Solidity of an Hexaëdrum may be had from *Conſect.* 3. *Definit.* 13.

III. A Dodecaëdrum conſiſts of twelve Quinquangular Pyramids, and an Icoſaëdrum of twenty Triangular ones, all the Vertex's or tops whereof meet in the Center of a Sphere that is conceived to circumſcribe the reſpective Solids, and conſequently they have their Altitudes and Baſes equal: Wherefore having found the Solidity of one of thoſe Pyramids, and multiplied it by the number of Baſes (in the one Solid 12, in the other 20) you have the Solidity of the whole reſpective Solids.

DEFINITION XXII.

BEsides these Definitions of the Regular Bodies, we may al- so form like Idea's of them from their Genesis, which particularly *Honoratus Fabri* has given us a short and ingenious System of, in his *Synopsis Geometrica*, p. 149. and the following.

I. Suppose an Equilateral Triangle ABD to be inscrib'd in a Circle (*Fig.* 44. *N.* 1.) whose Center is C, whence having conceived the Radii CA, CB, CD, to be drawn, imagine them to be lifted up together with the common Center C, so that the point C ascending Perpendicularly, at length you'l have the Lines EA, EB, ED, equal to the Lines AB, BD, DA, After this way there will be generated, or made a Space consisting of four Equal and Equilateral Triangles, which is call'd a *Tetraëdrum*. Hence we shall by and by easily demonstrate, the quantity of the Elevation CE, and the Proportion of the Diameter of the Sphere EF to be Circumscribed to the remaining part CF and so the reason of the Euclidean Genesis proposed *lib.* 13 *Prop.* 13.

II. Much like this, but somewat easier to be conceived, is the Genesis of the *Octaëdrum*, where by a mental raising of the Center C (*Fig.* 44. *N.* 2) of the Square ABDE inscribed in the Circle, together with the Semidiameters CA, CB, CD, CE until being more and more extended they at length become the Lines AF, BF, DF, EF, all equal among themselves, and to the side of the Square AB or BD; and its manifest, that by the like extension conceived to be made downwards to G, there will be formed eight equal and regular Triangles, which will all concur in the two opposite Points F and G. We might also deduce another Genesis of the Octaëdrum from a certain Section of a Sphere, and also give the like of a Hexaëdrum or Cube: but we have already given the easiest, of the one, *viz.* that which is also common to Parallelepipeds; and that of the other just now given is sufficient to our purpose.

CHAP. II.

Containing the Explication of those terms, which relate to the affections of the Objects of the Mathematicks.

DEFINITION XXIII.

EVery Magnitude is said to be either *Finite* if it has any bounds or terms of its Quantity; or *Infinite* if it has none, or at least *Indefinite* if those bounds are not determined, or at least not considered as so; as *Euclid* often supposes an *Infinite* Line, or rather perhaps, an *Indefinite* one, *i. e.* considered without any relation to its bounds or Ends: By a like distinction, and in reality the same with the former, all quantity is either *Measurable*, or such that some Measure or other repeated some number of Times, either exactly measures and so equals it, (which *Euclid* and other *Geometricians* emphatically or particularly call Measuring) or else is greater; or on the other side *Immense*, whose Amplitude or Extension no *Finite* Measure whatsoever, or how many times soever repeated, can ever equal: In the first Case, on the one Hand, the Measure (*viz.* which exactly measures any quantity) is called by *Euclid* an *aliquot Part* (*a*) or simply a *Part* of the thing measured: as *e. g.* the Length of one Foot is an aliquot Part of a Length or Line of 10 Foot. In the latter Case the Measure (which does not exactly measure any Quantity) is called an *Aliquant* Part, as a line of 3 or 4 Foot is an *Aliquant* part of a Line of 10 Foot. Now therefore, omitting that perplext Question, whether or not there may be an infinite Magnitude, we shall here, respecting what is to our purpose, deduce the following

(*a*) *lib.* 5. *Def.* 1.

CONSECTARY.

EVery Measure, or part strictly so taken, is to the thing Measured, or its whole, as Unity to a whole number, for that (which is one) repeated a certain number of times, is supposed exactly to measure the other.

DEFINITION XXIV.

IF the ſame Meaſure meaſures 2 different quantities (whether the one can exactly Meaſure the other or not) thoſe Quantities are ſaid to be abſolutely *Commenſurable*; but if they can have no common Meaſure; they are called *Incommenſurable*: Notwithſtanding which they both retain one to the other a certain relation of Quantity, which is call'd *Reaſon* or *Proportion*, as we ſhall further ſhew hereafter. In the mean while we have hence, as an infallible Rule to try whether Quantities can admit of a Common Meaſure or not, this

CONSECTARY.

THoſe Quantities are Commenſurable, whereof (*a*) one is to the other, either as Unity to an whole Number, or as one whole Number to another, for either one of them is the Meaſure of the other, as alſo of it ſelf, and then it is to that other as Unity to a Number by the *Conſect.* of the *preced.* or elſe they admit of ſome third Quantity for a common Meaſure, which will be to either of them ſeparately as Unity to ſome Number: therefore they are one to another as Number to Number.

(*a*) *Euclid lib.* 10. *Prop.* 5, 6, 7, 8.

DEFINITION XXV.

IF 2 Quantities of the ſame Kind, conſidered as Meaſures one of the other, being applyed one to the other, exactly agree, or are exactly equal every way, (as *e. g.* 2 Squares on the ſame common Side, or two Triangles whoſe Lines Angles and Spaces exactly agree and conicide) or at leaſt may be equally meaſured by a common Meaſure applyed to both) as *e. g.* a Square and an Oblong, or a Rhombus, or Triangle, each of whoſe Area's were 20 ſquare Inches, altho' they do not agree in Lines and Angles; the firſt may be called *Simply Equal*, and the other *totally equal, or equal as to their wholes*: But if one be greater and the other leſs, they are *Unequal*, and that which exceeds is called the *greater*, and that which is deficient the *leſs*, and that

par-

part by which the less is exceeded by the greater, in respect to the greater is call'd *Excess*, in respect to the less *Defect*, and by a common Name they are call'd the *Difference*. All which as they are plain and easy, so they afford us a great many self-evident Truths, which are used to be call'd Axioms, as these and the like

CONSECTARYS.

I. THe whole is greater than its Part, whether it be an Aliquot or aliquant Part.

II. Those Quantities which are equal to a third are equal betwixt themselves.

III. That which is greater or less than one of the equal Quantities is also greater or less than the other.

IV. Those Quantities which, being applyed one to the other, or placed one upon the other, either really or mentally, agree; may be esteemed as *totally equal*: And on the Contrary,

V. Those Quantities which are *totally equal* will agree, *&c*. To which might be added several others which we have already made use of and supposed as such in the preceding *Definitions*.

DEFINITION XXVI.

THere are moreover *Addition*, *Subtraction*, *Multiplication* and *Division*, which are common affections of all Quantities as well as of Numbers. *Addition* is the Collection of several Quantities (for the most part of one kind) into one total or Sum; which is either done so, that the whole (which is commonly called the *Sum* or *Aggregate*) obtains a new Name, or else by a bare connexion of the Quantities to be added by the Copulative *and*, or the usual Sign + (*i. e. plus* or more) as for Example 2 Numbers . . . and : . . . (suppose 3 and 4) added together make the Sum . . . . (*i. e.* 7, or which is the same thing 3+4;) and this Line ——— added to this other——— ——gives the Sum——————— which is nothing but the 2 Lines joyn'd, or taken together. But now if we would treat of these Lines, or any other 2 Quantities to be added, more generally; by calling the first a (a) and the latter (b) we may fitly write their Sum $a+b$.

SCHOLIUM.

HAving thus explained the Term of *Addition*, these and the like Axioms emerge of themselves: *If to equal Quantities you add Equal the Sum will be Equal; but if to Equal you add unequal the Aggregate will be unequal*, &c. Moreover it may not be amiss to admonish the *Tyro* of these 2 things. 1. In Addition may be seen the vast usefulness of that very Ingenious tho' familiar Invention mentioned in *Definit*. 3. for hereby we may collect into one Sum not only Tens, and Hundreds, but Thousands, Millions, Myriads, as tho' they were only Units; which we will Illustrate by an Example.

DEFINITION XXVII.

SUbtraction is the taking one Quantity from another (of the same kind;) which is so performed that either the remainder obtains a new Name, or by a bare separation of the *Subtrahend* by the privative Particle less, or the usual *Sign* — which stands for it, as *e. g.* . . . or three being subtracted from . . . . . or 7, the remainder or difference is . . . . or 4 and this Line ——— Subtracted from that ———— leaves ——— Now if we would signify this more generally either of these Lines, or the Number above, or any 2 Quantities whatsoever that are to be Subtracted one from the other, by naming the first (a) and the latter (b) we shall have the remainder $a - b$. Herein are evident these and the like Axioms: *If from equal Quantities you Subtract Equal ones, the Remainders or Differences will be equal.* Here it will be worth while to take notice of, from this and the *preced. Definit.* the following

CONSECTARYS.

I. IF a negative Quantity be added to it self considered as positive (as -3 to $+3$ or $-a$ to $+a$) the Sum will be 0: for to add a Privation or Negative is the same thing as to Subtract a Positive, wherefore to join a Negative and Positive together, is to make the one to destroy the other.

II.

II. If a negative be ſubtracted from its poſitive ($-a$ from $+a$) the remainder will be double of that poſitive ($+2a$) for to ſubtract or take away a privation or negative, is to add that very thing, the privation of which you take away; for really that which in words is called the *addition of a Privation*, is in reality a Subtraction, and a *ſubtraction* of it, is really an *addition*; and what is here call'd a *Remainder*, is indeed a Sum or Aggregate; and what is there call'd a *Sum*, is truly a *Remainder*. Thus,

III. If the poſitive Quantity ($+a$) be taken from the privative one ($-a$) the remainder is double the privative one ($-2a$) ſince, taking away a poſitive one, there neceſſarily ariſes a new Privation which will double that you had before. Hence,

IV. You have the Original of the Vulgar Rules in Literal Addition and Subtraction: *If the Signs of the unequal Quantities are different, in the room of Addition you muſt ſubtract, and in room of Subtraction add, and to the ſum or remainder, prefix the Sign in the firſt place of the greateſt, in the next of that from which you Subtract: but if the Signs are both the ſame, and the greateſt quantity to be ſubtracted from the leſs, you muſt, on the contrary, ſubtract according to the natural Way, the leaſt from the greater, and prefix the contrary Sign to the Remainder:* Which Rules you may ſee Illuſtrated in the following Examples:

Addition	*Subtraction.*	
$4b - 2a$	from $2a + b$	from $3a + 2b$
$3b + 5a$	Subſt. $a - b$	Subſt. $2a + 3b$
$7b + 3a$	R. $a + 2b$	R. $a - b$

NOTE.

☞ Inſtead of the Authors 4th Conſect. as far as it relates to Subtraction, which may ſeem a little perplext, take this general Rule for Subtraction in Species, *viz.* Change all the Signs of the lower Line, or Subtrahend, and then add the Quantities, and you have the true Remainder.

SCHOLIUM.

IN this Literal Subtraction, we have not that conveniency which the invention of Vulgar Notes supplies us with, that from the next foregoing Note we may borrow *Unity*, which in the following Series goes for 10, *&c.* This is done in Tetractycal Subtraction only with this difference, that an Unite here borrowed goes only for 4. That the easiness of this Operation may appear, we will add one Example, wherein from this number, —— 12320023110232
you are to subtract this, 3210123321223

3103233223003

Whereever therefore the inferiour Note is greater than the superiour one, the facility is much greater here than in common Subtraction, because never a greater number than 3 is to be subtracted out of a greater, than 4 and 2: but if the inferiour number be greater than the superiour, you borrow unity from the left hand, which is equivalent to 4; the rest is perform'd as in common Subtraction.

DEFINITION XXVIII.

MUltiplication, generally Speaking, is nothing else but a Complex or manifold Addition of the same quantity, wherein that which is produced is peculiarly call'd the *Product*, and those quantities by which it is produced, are called the *Multiplicand* and the *Multiplier*: The first denotes the Quantity which is to be multiplied, or added so many times to its self; and the other the Number by which it is to be multiplied, or determins how many times it is to be added to it self. The same terms are applyed moreover to Lines and other Quantities. But here are two things to be chiefly noted; 1. That the Multiplication of one number by another, or of a Line by a Line, may be considered as having a double Event; for the Product may be either of the same or a different kind, as, *e. g.* when . . . . 4 is multiplied by 3 . . . the product may be considered either as a Line, thus, ... or as a

a Plane Surface in this Form,

. . . .
. . . .
. . . .

Whence it is also named a Plane Number, and the product is conceived to be formed by the motion of an erect Line AB, consisting of 3 equal parts, along another BC, consisting of 4 equal parts, and conceived as lying along. So also the Multiplication of Lines (*e.g.* of the Line A —— B by the Line B———— C) may be conceived to be so performed, that the Product also shall be a Line, *e. g.* C——— ——— ———D (concerning the usefulness of which Multiplication in *Geometry*, we shall have occasion to speak more hereafter;) or so, that the Product shall be a Plane or Surface, arising from the motion of the erect Line AB, along AC, conceiv'd as lying along; as we have already shewn. But as for the most part these Planes so produced are called *Rectangles*, if the Lines that form them are unequal; but if they are equal they are call'd *Squares*, (otherwise the Powers of the given Quantities;) and in this case the Lines that form them are called *Square Roots*; so also if those Planes are multiplied again into a third Quantity (as either a Line or a Number) there will arise *Solids*, and particularly if that third Quantity be the Root of the Square, the Product is called a *Cube*, &c. The other thing to be noted is, That both these ways of Multiplying either Numbers or Lines, are expressed by a very compendious, tho arbitrary way, of Notation, *viz.* by a bare *Juxtaposition* of the Letters which denote such and such Species of Quantities, as, *e. g.* if for the forementioned Number or Line AB we put a, and for BC b, the Product will be ab; or if the Efficients are equal, as a and a the Square thence produced, will be aa or a^2; and if this Square be further multiplied by its Root a, then the Cube thence produced will be aaa or a^3, &c. Which being premised, you have these following

CONSECTARIES.

I. IF a Positive Quantity be multiplied by a Positive one, the Product will be also Positive; since to multiply is to repeat the Quantity according as the Multiplier directs: Wherefore to multiply by a Positive Quantity, is to repeat the Quantities positively; as on the other side, to multiply by

a

a Privative, is ſo many times to repeat the Privation of that Thing: Which we ſhall ſhew further hereafter.

II. Equal Quantities (a and a) multiplied by the ſame (b), or contrariwiſe, will give equal Products ($a\,b$ and $a\,b$ or $b\,a$).

III. The ſame Quantity (z) multiplied by the whole Quantity ($a+b+c$) or by (a) all its parts ſeparately, will give equal Products. Alſo

IV. The whole ($a+b$) whether it be multiplied by (b) it ſelf, or by its parts ſeparately, will give equal Products.

(a) *Eucl. lib.* 2. *prop.* 1.
(b) *lib.* 2. *prop.* 2.

SCHOLIUM I.

THe Vulgar Praxis of Numeral Multiplication, is founded on theſe two laſt Conſectarys, as *e. g.* to multiply 126 by 3; you firſt multiply 6 by 3, then 2, *i. e.* 20 by 3, then 1, *i. e.* 100 by the ſame, and then add each of thoſe partial Products into one Sum: In like manner being to multiply 348 by 23, you firſt multiply each Note of the Multiplicand by the firſt of the Multiplier (3) and then by the ſecond (2) (*i. e.* 20) &c. which is to be done likewiſe after the ſame manner in Tetractical Multiplication; only in this latter, which is more eaſie, you have nothing to reſerve in your mind, but all is immediately writ down, (which might alſo be done in Vulgar Multiplication) as may be ſeen by this Example underneath, as alſo the great eaſineſs of this ſort of Multiplication, beyond the common way, becauſe there is no need of any longer Table than that we have ſhewn page 7.

```
      3 2 1 0 3 3
            1 2 3
  ---------------
    2 1 2 3 2 1 2
    1 1 2 1 0 1
  1 2 0 0 1 0
  3 2 1 0 3 2
          2
  ---------------
  1 2 0 0 2 0 3 2 2
```

SCHO-

SCHOLIUM II.

It is manifeſt from what we have ſaid,

I. IF the Baſe of a Parallelogram be called (b) and its Altitude a, its Area may be expreſſed by the Product ab, by *Conſ.* 7. *Definit.* 12.

II. If the Baſe of a △ be b or eb, and its Altitude a its Area will be half ab or half eab, by *Conſectary* 8. of the ſame *Definition.*

III. If the Baſe of a Priſm or Parallelepiped or Pyramid be half ab or ab, and its Altitude c, the ſolid Contents of that Priſm will be half abc, and of the Parallelepiped abc, by *Conſect.* 3 & 4. *Def.* 16. and of the Pyramid $\frac{1}{6}abc$, by *Conſ.* 3. *Def.* 17.

DEFINITION XXIX.

DIviſion, in general, is a manifold or complicated Subtraction of one quantity (which is called the *Diviſor*) out of another (which is called the *Dividend*) whoſe multiplicity, or how many times the one is contained in the other, is ſhewn by another quantity ariſing from that Diviſion, which is therefore called the *Quote* or *Quotient.* Here alſo the *Diviſor* is of the ſame kind with the *Dividend*, or of a different kind, *e. g.* of the ſame kind if the product . . . . (12) be divided by (3) whence you'l have the Quotient . . . . (4) or dividing the aforementioned Line CD by the Line AB you'l again have the Line BC; but of a different kind, if the plane number a-

. . . .

bove found . . . . or the Rectangle ABCD be divided by

. . . .

a Retroduction, or a moving backwards again the erect Side AB, by whoſe motion the Rectangle was firſt formed, that ſo the Line BC may remain alone again. But both theſe kinds of Diviſion as they have their peculiar Difficulties in Arithmetick and Geometry, which we ſhall further elucidate in their proper places; ſo they may be univerſally and very eaſily performed in Species (or by Letters) which will be ſufficient to our preſent purpoſe; or by a bare ſeparation of the Diviſor from the Dividend, if it be actually therein included; or by placing

placing the Divisor underneath the Dividend with a Line between. Thus if $a\,b$ be to be divided by (b) the Quotient will be a; if by a, the Quotient will be (b); but if a or $a\,b$ be to be divided by c which Letter since it is not found in the Dividend, cannot be taken out of it) the Quotients are $\frac{a}{c}$ and $\frac{a\,b}{c}$ *i. e.* a or $a\,b$ divided by c, after the same manner as if 2 were to be divided by 3; which Divisor, since it is not contained in the Dividend, is usually placed under it, by a separating Line thus, $\frac{2}{3}$, 2 divided by 3.

SCHOLION.

HOW difficult Common Division is, especially of a large Dividend by a large Divisor, is sufficiently known: but how easily it is performed by Tetractical Arithmetick, we will barely bring one Example to shew. If the Product found in *Schol.* 1. of the preceding *Definition*, 1200 203 22 be again to be divided by its Multiplier 1 3 3, it may be performed after the usual way, but with much more ease, as the following Operation will shew; or according to a particular way of *Weigelius*, by writing down the Divisor, and its double and triple, in a piece of paper by it self, after this way:

1 2 3	3 1 2	1101
Divisor,	*Double*,	*Triple*.

and then moving that piece of Paper to the Dividend, note, which of those three Numbers comes nearest to the first Figures of the Dividend; for that barely subtracted gives the Remainder, and will denote the Quotient to be writ down in its proper place; as the operation itself will shew better than any words can.

$$\begin{array}{c}
\cancel{1}\cancel{1}\cancel{1}\cancel{1}\cancel{1} \\
\cancel{1}\cancel{3}\cancel{2}\cancel{2}\cancel{2}\cancel{2} \\
\cancel{3}\cancel{2}\cancel{3}\cancel{0}\cancel{1}\cancel{1}\cancel{1} \\
\cancel{1}\cancel{2}\cancel{0}\cancel{0}\cancel{2}\cancel{0}\cancel{3}\cancel{2}\cancel{2} \\
\cancel{1}\cancel{2}\cancel{3}\cancel{3}\cancel{3}333 \\
\cancel{1}\cancel{2}\cancel{2}\cancel{2}\cancel{2}\cancel{2}2 \\
\cancel{1}\cancel{1}\cancel{1}\cancel{1}
\end{array}
\Big(321032$$

Thus

Thus after *Weigelius*'s way:

$$\begin{array}{r|l|l} \cancel{1} & & \\ 2 & \cancel{1} & \\ 33 & \cancel{0}.\cancel{1}\cancel{1} & \\ \cancel{1}2\cancel{0}\cancel{0} & 2\cancel{0}3.22 & 321032. \end{array}$$

DEFINITION XXX.

E*Xtraction of Roots* is a Species of Diviſion, wherein the Quotient is the Root of the given Square or Cube, *&c.* But the Diviſor is not given, neither is it all along the ſame (as it is in Diviſion) but muſt be perpetually found, and they are ſeveral. And as the Squares of Simple Numbers 1, 2, 3, &c. *viz.* 1, 4, 9, 16, *&c.* and their Cubes 1, 8, 27, 64, *&c.* may be had immediately out of a Multiplication Table, as alſo their Roots, without any further trouble; and likewiſe in Species, as the Roots of the Square *aa* or a^2, or of the Cube *aaa* or a^3, are without doubt (*a*); ſo if the Square Root be to be extracted out of *d e*, or the Cube Root out of *f g m* (becauſe the letters are different, and no one can be taken for the Root) the Square Root is commonly noted by this Sign $\sqrt{de}$, the Cube Root by this $\sqrt{C}$, or $\sqrt[3]{fgm}$, &c. as alſo in Numbers that are not perfect Squares (as *e.g.* 2, 3, 5, 6, 7, 8, 10, 11, 15, 17, 19, *&c.*) we can no otherwiſe expreſs the Square Roots, then after this manner $\sqrt{2}$, 7, $\sqrt{19}$, *&c.* and in thoſe that are not perfectly Cubical (as all between 1, 8, 27, 64, *&c.*) we can only expreſs their Cube Roots after ſome ſuch manner, $\sqrt{c.\,7}$, or $\sqrt[3]{7}$. $\sqrt{c.61}$, or $\sqrt[3]{61}$ *&c.* Which forms of Roots in ſpecious Computation, we call *Surd Quantities*, in Vulgar Arithmetick *Surd Numbers*, *i. e.* ſuch as cannot be perfectly expreſſed by any Numbers; altho we have Rules at hand to determine their Values nearer and nearer *ad Infinitum*.

Theſe Rules accommodated to Square and Cube Numbers, *&c.* which otherwiſe are more difficult to be comprehended, appear plain and eaſie to him, who multiplies a Root expreſſed by

by 2 Letters (called therefore commonly a Binomial) first Quadratically, then Cubically, *&c.* For he will have as

CONSECTARYS.

I. THE Square of any assumed Root, as also, *Prop.* 4. *lib.* 2. *Eucl.* and at the same time a general Rule for Extracting the Square Root, all expressed in these few Notes:

$$aa + 2ab + bb.$$

II. The Cube of the same Root, a New Theorem, and at the same time a Rule for Extracting any Cube Root, contained in this Theorem:

$$a^3 + 3aab + 3abb + bbb.$$

SCHOLIUM I.

WHich that we may more plainly shew, especially as far as it relates to the Rules of Extraction, consider, 1. That the Root of the Square $aa + 2ab + bb$ is already known (for we assumed for the Root the Quantity $a + b$) so that now we are to see which way this Root is to be obtain'd out of that Square by Division. It will presently appear, that the first Note of the Root a, will come out of the first part of the Square aa, and the other part b must be obtain'd out of the remainder $2ab + bb$; and so as there are 2 Notes of the Root, the Square must be distinguish'd as it were into 2 Classes, each of which gives a particular note of the Root. Then it is manifest, that the first Note of the Root (a) may be obtain'd out of the Square aa by a simple Extraction. Now it is evident, if I would have by Division the other Note of the Root, the next following part of the remaining Classis must be divided by $2a$, the double of the Quotient just now found, and that nothing should remain after this Division (for now we have the whole Root $a + b$) you must not only subtract the Product of the Divisor and this new Quotient, but also the Square of this new Quotient: Which is the Vulgar Method and Rule for the Extraction of Square Roots taught in common Arithmetick.

Likewise

Likewise if you would extract the Root of the above-mentioned Cube, which we already know, having formed it from $a+b$, it is manifest, that the first Note of the Root a will come out of the first part of the Cube a^3, and the other b, must be obtain'd out of the remainder $3abb+bbb$, and so, as there are two Notes of the Root, the Cube must be distinguish'd, as it were into two Classes, each of which will give a particular Note of the Root. Now it is manifest, the first Note of the Root a is obtained by simple Extraction of the Root out of the Cube aaa. It is moreover evident, if I would have by Division the other Note of the Root b, the next remaining part must be divided by $3aa$ (the triple Square of the precedent Quotient, or thrice the precedent Quotient multiplied by it self) and, that nothing should remain after this division (for now we have the whole Cube Root $a+b$) you must not only subtract from the remaining Dividend the Product of the Divisor, and the new Quotient ($3aab$, but also the Product of the Square of the new Quotient, and thrice the precedent Quotient ($3abb$) and moreover the Cube of that new Quotient b^3: Which is the Method of extracting Cube Roots in Vulgar Arithmetick.

SCHOLIUM II.

FRom what we have said you have also the Reason of another rule in Arithmetick which teaches how to approach continually nearer and nearer to the Square and Cube Roots of numbers that are not exact Squares and Cubes; *viz.* by adding to the given Number perpetually new Classes and Cyphers or o's, two at a time, to the Square, and three to the Cube, and so continue on the operation as before; which will add Decimal Parts to the Integrals before found; and the next operation (if you add a second Classe of Cyphers) will exhibit Centesimal Parts, and so on *ad Infinitum*. For Example, If I would have the Square Root of 2 pretty near, I can assign no nearer whole Number than 1. But by adding a new Classe of 2 Cyphers, *i.e.* multiplying the given Number by 100 (whereby the Root is multiplied by 10) you'l have 14, nearly the Root of 200, that is, $\frac{14}{10}$ or $1\frac{4}{10}$ much nearer the Root of 2 than the former; and thus you may always come nearer and nearer *ad Infinitum*, but never to an exact Root. For if you could have the

the exact Root of 2 ,or 3, or 5, &c. in any Fraction whatsoever, that Fraction must be of such sort, that its Numerator and Denominator being squared, the Fraction thence arising must exactly equal 2 or 3, or 5, &c. that is, its Numerator must be exactly double, or trible, or Quadruple, &c. of the Denominator; which can never be, because both are Squares, and in a Series of Squares no such thing can happen. Hence you have these

CONSECTARYS.

III. THat it is a certain mark of Incommensurability, if one quantity is 1, and other the $\sqrt{2}$, or $\sqrt{3}$, or $\sqrt{5}$, &c.

IV. That these sorts of Quantities are notwithstanding Commensurable in their Powers, *i. e.* their Squares are as 1 and 2, or as a number to a number.

V. Those Quantities which are to one another, as 1 and $\sqrt{\sqrt{2}}$, or as $\sqrt{2}$ and the $\sqrt{\sqrt{3}}$ are incommensurable in Power also. Which being rightly understood, you may easily comprehend several (*a*) Propositions of *lib.* 10 *Eucl.* especially after some few things premised concerning Reason and Proportion.

SCHOLIUM. III.

FRom what we have shewn may easily be concluded, that to any proposed Quantity whatsoever, which *Euclid* calls (*b*) *Rational*, and for which we may always put I, there may be several others both commensurable and incommensurable, and that either simply or in power so; those which are commensurable to a Rational given Quantity, either Simply or only in Power (which, *e.g.* are to 1, as 2, 3, 4, &c. $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c. or as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$, $\sqrt{\frac{1}{2}}$, $\sqrt{\frac{1}{3}}$, $\sqrt{\frac{1}{4}}$, &c.) are called also *Rational*: but those which are Incommensurable both ways (*i.e.* both simply and in power) as ($\sqrt{\sqrt{2}}$, $\sqrt{\sqrt{\frac{1}{3}}}$, &c.) are called *Irrational*. In like manner the Square of a given Rational Quantity (as I) is called Rational, and Quantities commensurable to it (as 2, 3, 4, 5, &c. $\frac{1}{2}$, $\frac{1}{3}$, &c. □) are called also *Rational*; but incommensurable ones ($\sqrt{\sqrt{2}}$, $\sqrt{\sqrt{5}}$, &c.) *Irrational*, and the Sides and Roots of these more *Irrational*.

(*a*) *Eucl. l.* 10. *Prop.* 9, 10, 11, 12, 13 &c.

(*b*) *lib.* 10, *Def.* 5, 6, 7, 8, 9, 10, 11.

DEF.

DEFINITION XXXI.

ANy Quantities whatſoever of the ſame kind, whether commenſurable or incommenſurable, equal or unequal, admit of a twofold reſpect or relation of their magnitude, one whereof, when only the difference or exceſs of one above another is reſpected (as 10 which is 3 more than 7) is called an *Arithmetical Reaſon* or *Reſpect*; the other, wherein reſpect is rather had to the Amplitude, whereby one is contained once, or a certain number of times in the other (as 3 is contained thrice in 10 and $\frac{1}{3}$ part more) is called *Geometrical Reaſon*, or by way of Emphaſis, only *Reaſon*, and by others *Proportion*; and this Reaſon or Proportion, if the leſs is exactly contained a certain number of times in the greater (as 3 in 6, or 4 in 12) is generally called, on the part of the greateſt term, *Multiple*, and on part of the leſs, *Submultiple*, and particularly in the firſt Example *double*, when 6 is taken in reſpect to 3, and *ſubduple* when 3 is taken in reſpect to 6; in the other *triple* and *ſubtriple*, *&c.* If the leſs be contained in the greater once or more times, unity only remaining over and above (as 3 in 4 and 4 in 9) the *Reaſon* or *Proportion* is called *Superparticular* and *Subſuperparticular*, and is noted by the terms *Seſqui* & *Subſeſqui*, joyning the ordinal Name of the leſſer Term; as the the Reaſon of 4 to 3 is called *Seſquitertian*, and contrariwiſe *Subſeſquitertian*; the Reaſon of 9 to 4 is called double *Seſquiquartan*, and contrariwiſe *Subduple Subſequiquartan*, &c. If laſtly, the leſs be contained in the greater once, or a certain number of times, ſeveral units remaining over and above, it is commonly called *Superpartient* Reaſon and is expreſſed by the word *Super* or *Subſuper*, joyned with the adverbial Name of the remaining Parts, and the ordinal Name of the leſſer Term; thus, *e.g.* the Reaſon of 7 to 4, is called *Supertriquartan*, 12 to 5 double Superbiquintan, *&c.* but when the Quotient ariſes by the diviſion of the greater term by the leſs, and is commonly expreſſed in the ſame words, it is alſo commonly called by the name of *Reaſon* (*e.g.* 2 is the name of the Reaſon of 6 to 3, $2\frac{1}{4}$ of 9 to 4, or contrariwiſe, *&c.*) as alſo the quotient ariſing by diviſion of the Conſequent by the Antecedent (as $\frac{1}{2}$ in the firſt caſe, $\frac{4}{9}$ in the latter) by which name the antecedent Term of the Reaſon being multiplied, produces its Conſequent; which is evident by naming any Reaſon

e or *i*, or *o*, *&c.* Thus if the antecedent Term be called *a* or *b* &c. the Consequent may be rightly call'd *ea* or *e b*, *oa* or *ob*, &c. and because in an Arithmetical Relation we only respect the excess of the first above the following, or of the following above the foregoing (which may be called x or z) if the antecedent (which may be called a or b) be less, the consequent may properly be called $a+x$ or $b+z$; but if it be greater, the other will be $a-x$ or $b-z$.

CONSECTARYS.

I. WE may hence readily infer, that if the Diameter of any Circle be called a the Circumference may be called $e\,a$, (for whatever the proportion is between them, it may be expressed by the Letter e) and the Area, according to *Consectary* 2. *Definition* 15, will be $\frac{1}{4}\,e\,a\,a$.

II. If for the Base of any Cylinder or Cone you put $\frac{1}{4}\,e\,a\,a$ and the Altitude (b) the Solidity of that Cylinder may be rightly expressed by $\frac{1}{4}\,e\,a\,a\,b$, by *Consect.* 5. *Definit.* 16, and of the Cone by $\frac{1}{2}\,e\,a\,a\,b$, by *Consect.* 4. *Definit.* 17.

DEFINITION XXXII.

AS the Identity (or sameness) of several Geometrical Reasons used to be called *Geometrical Proportionality*, or emphatically *Proportion*; so the similitude (or likeness) of several *Arithmetical Reasons*, is deservedly call'd *Arithmetical Proportionality*, or by a particular Name *Progression*; and consequently those *Progressionals*, or *Arithmetical Proportionals*, which exceed one another by the same difference, either uninterruptedly or continually, as 2, 5, 8, 11, 14, *&c.* ascending, or 30, 28, 26, 24, 22, 20, &c. descending; or interruptedly, as 2 and 5, 7 and 10, 11 and 14, *&c.* ascending; or 30 and 26, 24 and 20, 18 and 13, *&c.* descending: For which, and all other in what case soever, we may universally put this (or such like) continual Progression, *v. g.* a, $a+x$, $a+2x$, $a+3x$, *&c.* ascending; or a, $a-x$, $a-2x$, $a-3x$, *&c.* descending, but in an interrupted Progression, *v. g.* b and $b+z$ and c and $c+z$, c and $c-z$, d and $d-z$, *&c.* descending. Whence you have this

CON-

CONSECTARY.

ANY Difference being given, the following Terms of the Progression, continually proceeding from the first assumed or given one, may be found; as also several Antecedents that interruptedly follow the given or assumed ones, *viz.* by adding or subtracting the given Difference to or from the former Terms to find the latter.

DEFINITION XXXIII.

IN like manner, since *Reasons* are said to be the same, which have the same Denomination of Reason, those quantities will be proportional which continually ascend by the same denomination of Reason, as 2, 4, 8, 16, 32, 64, *&c.* or descend, as 81, 27, 9, 3, 1. there by the Denomination of the Reason 2, here 3; or that ascend interruptedly, as 2, 4; 3, 6: 5, 10, &c. or descend, as 40, 10; 28, 7; 20, 5; 8, 2, *&c.* Whence you have these

CONSECTAYS.

I. HAving two Terms given, or only one with the Denomination of the Reason (*e.g.* the Term 2 with the Denomination of the Reason 3, or universally the first Term a with the Denomination of the Reason e) it will be easie to find as many more Terms of the Geometrical Progression or Proportion as you please, *viz.* by always multiplying the Antecedent by the Denomination of the Reason, that you may have 2, 6, 18, 54, *&c.* or a, ea, e^2a, e^3a, *&c.* in continued, or 2 and 6, 4 and 12, 5 and 15, *&c.* and $a\ e\ a$, $b\ e\ b$, $d\ e\ d$, *&c.* in discontinued or interrupted Proportion.

Thus having rightly understood what we have said in this 33 and 31 *Definition*, there will follow these *Corollarys* as so many Axioms.

II. That equal Quantities have the same proportion to the

 same

ſame Quantity (*α*) and the ſame has the like to equal Quantities.

III. But a greater quantity has a greater Reaſon to the ſame (*β*) than a leſs, and the ſame has a greater proportion to a leſs Quantity than to a greater.

IV. On the contrary, thoſe that have the (*γ*) ſame proportion to the ſame quantity, and that likewiſe the ſame to them are equal.

V. But that which bears a (*δ*) greater proportion to the ſame is greater; but that to which the ſame bears a greater proportion is leſs.

VI. Proportions equal to one third (*ε*) are alſo equal among themſelves, *&c.*

DEFINITION XXXIV.

HEre remain two things to be taken notice of; Firſt that If any whole (quantiy) be ſo divided into two equal parts (*α*) that the whole, the greater part and the leſs are in a continual proportion; that (whole) is ſaid to be cut in extreme and mean Reaſon. 2. In a continual Series of that kind of Proportionals (*e. g.* 2. 4. 8. 16. 32, *&c.* or a, ea, e^2a, e^3a, e^4a, *&c.*) the Reaſon of the firſt Term to the third (*β*) (2 to 8, or a to e^2a) is particularly called *Duplicate*, and to the 4th (16 or e^3a) *Triplicate*, &c. of that Reaſon which the ſame firſt Term has to its ſecond, or any other antecedent of that Series to its Conſequent: But generally theſe Duplicate and Triplicate Reaſons, *&c.* as others alſo of the firſt Term to the third or fourth of Proportions continually cohering together, (whether they are the ſame as in the foregoing Examples, or different as theſe, 2, 4, 6, 18, or a, ea, eia, $eioa$, *&c. viz.* if the name of the firſt Reaſon be e, of the ſecond i, of the third o, *&c.*) I ſay, the Reaſons of the firſt Term (2 or a) to the third (6 or eia) or to the 4th (18 or $eioa$) are ſaid to be compounded of the continual intermediate Reaſons.

(*α*) *Eucl. lib. Prop.* 7.
(*β*) *prop.* 8.
(*γ*) *prop.* 9.
(*δ*) *prop.* 10
(*ε*) 16.

(*α*) *Eucl. Definit.* 3. *lib.* 6
(*β*) *Eucl. Definit.* 10. *l.* 5.

Now from our general Example, what *Euclid* ſays, is manifeſt,

CO

CONSECTARY I.

THat the denomination of a compounded Reaſon ariſes from the Multiplication of the denominations of the given Simple (α) Reaſons; as the denomination of the reaſon compounded of both (*viz.* *a* to *eia*) is produced by multiplying the denomination of the firſt Reaſon *e* by the denomination of the ſecond Reaſon *i*, and the denomination of the Reaſon compounded of the three (*viz.* *a* to *eioa*) is produced by the denomination of the firſt Reaſon *e*, multiplied by the denomination of the ſecond Reaſon *i*; and the Product of theſe by the denomination of the third Reaſon *o*, &c.

CONSECTARY II.

SO that it is very eaſie after this way, having never ſo many Reaſons given, whether continued (as 2 to 3, 3 to 6, or *a*, *ea*, *eia*,) or interrupted or diſcrete (as 2 to 3, and 5 to 10, or *a* to *ea*, and *b* to *i b*) to expreſs their compounded Reaſon: In the firſt caſe it eaſily obtain'd by the bare omiſſion of the intermediate Term or Terms (2 to 6, or *a* to *e i a*;) and in the other by multiplying firſt of all the Names of the compounding Reaſons among themſelves ($1\frac{1}{2}$ and 2, *e*. and *i*.) and by the Product (3 or *e i*) as the name of the Reaſon compounding the firſt Term (2 or *a*) that you may have the other 6 or *eia*) or (if any one had rather do ſo in this latter caſe) by turning the diſcrete or interrupted Reaſons into continued ones, by making as 5 to 10 in the ſecond Reaſon, ſo is the Conſequent of the firſt 3 to 6, or as *b* to *i b*, ſo *e a* to *e i a*,) and then by referring the firſt 2 to the third 6, or the firſt *a* to the third *e i a*, &c. In a word therefore, any Duplicate Reaſon may be appoſitely expreſſed by *a* to e^2a, and Triplicate by *a* to e^3a, the one immediately diſcernible by a double, the other by a triple Multiplication into itſelf; as you may alſo commodiouſly, and denote others compounded, *e. g.* of 2 by *a* to *e i a*, of 3 by *a* to *e i o a*, &c.

(α) *Eucl. l.* 6. *Def.* 5.

SCHOLIUM.

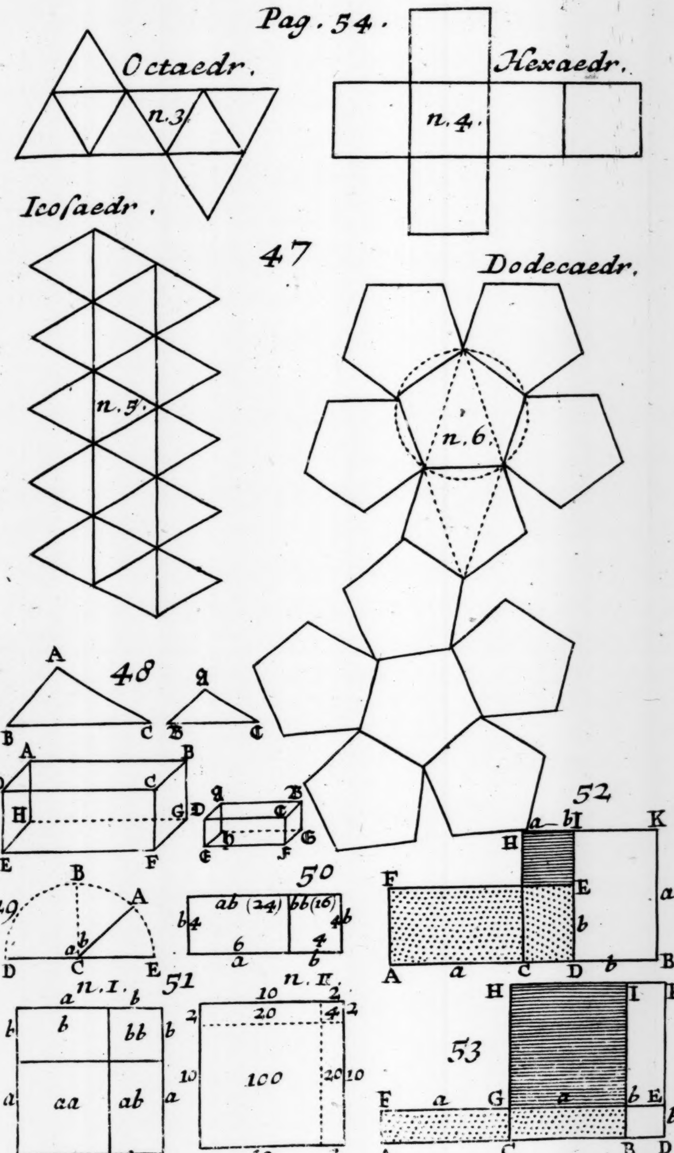
WE will here advertise the Reader, that tho the Names of duplicate & triplicate Reasons, &c. are chiefly appropriate to Geometrical Proportionality, yet the Moderns have also accommodated them to Arithmetical also; as *e. g.* That *Arithmetical Progression* is called *Duplicate*, whose Terms are the Squares of Numbers Arithmetically Proportional (*e. g.* 1, 4, 9, 16, 25, &c.) and *Triplicate*, whose Terms are Cubes, (&c. as 1, 8, 27, 64, &c.

DEFINITION XXXV.

AND now at length we may understand what Magnitudes Geometers particularly call *like*, or *similar*. Whereas in General one number may be said to be like another, one right Line to another, one obtuse Angle to another, a Triangle to a Triangle, and the like; but an Acute Angle is not like an Obtuse one, nor a Triangle like a Parallelogram, or a right Line like a Curve one; or a Square like an Oblong, *&c.* Yet among those Figures which may after that rate in general be said to be like, there is notwithstanding a great deal of dissimilitude; therefore in a strict Sense we call only those Right Lined Figures *similar* or *like* (*a*) which have each of their Angles respectively equal to each of the other (as A and 𝔄) B and 𝔅 C and ℭ, *&c.* *Fig.* 48.) and the Sides about those equal Angles Proportional, *viz.* as BA to AC, so 𝔅𝔄 to 𝔄ℭ, *&c.* (*b*) and among Solid Figures those are said to be Similar, each of whose Planes are respectively Similar one to the other, and equal in number on both sides; as, *e. g.* the Plane AC is similar to the Plane 𝔄ℭ, and CG to ℭℌ, *&c.* and six in number on both sides.

Book I

Pag. 54.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Book I.

Section II.

Containing several Propositions demonstrated from the foregoing Foundations.

CHAP. I.

Of the Composition and Division of Quantities.

Proposition I.

THE Sum and Difference of two unequal Quantities added together, make double of the greatest.

Demonstration.

Suppose *a* be the greatest, *b* the least, then will their Sum be $a + b$

And their Difference $a - b$

Their Sum $2a$, by *Consectary* 1. *Definition* 27, Q. E. D.

CONSECTARY.

HEnce by a bare Subsumption (α) you have the truth of *Consect.* 1. *Definit.* 8. that 2 unequal contiguous Angles on the same Right Line, ACD and ACE (*Fig.* 49.) *i. e.* if we call the Right Angle BCD or BCE (*a*) and the difference between the one and the other (*b*) $a+b$ and $a-b$, make $2a$, *i. e.* are equal to 2 right ones.

(α) *Eucl. lib.* 6. *Definit.* 1.
(β) *lib.* 11. *Def.* 9.
(α) *Eucl.* 13. *l.* 1.

Proposition II.

IF the Difference of two unequal Quantities be ſubtracted from their Sum, the Remainder will be double of the leaſt.

Demonſtration.

If from the Aggregate or Sum	$a + b$
You ſubtract the Difference	$a - b$
The Remainder will be	$0 + 2b$.

by *Conſectary* 2. *Definition* 27. Q. E. D.

Proposition III.

BUt if the Sum or Aggregate be ſubtracteed from the Difference, the remainder is ſo much leſs than nothing, as is the double of the laſt Quantity.

Demonſtration.

For if from the Difference	$a - b$
You ſubtract the Sum or Aggregate	$a + b$
The Remainder will be ——	$0 - 2b$,

by *Conſectary* 3 of the aforeſaid *Definition*. Q. E. D.

Proposition IV.

IF a Poſitive Quantity be multiplied by a Negative one, or contrariwiſe, the Product will be a Negative Quantity.

Expoſition.

If a—b *is to be multiplied by* c; *it is certain, that* a *multiplied by* c, *makes* a c *a Poſitive Quantity, by* Conſect. 1. Definit. 28. *Moreover* b *by the ſame* c (*a Negative by a Poſitive*) *will make* —bc; *and ſo the whole Product of* a—b *by* +c, *will be* ac—bc.

Demon-

Demonstration.

Suppose $a-b=e$; therefore ec will be $=$ to the Product of $a-b$ by c: and since $a-b$ is $=e$ by the Hypoth. adding on both sides b, you'l have $a=e+b$. by *Schol. Definit.* 26. and multiplying both sides by c, $ac=ec+bc$, by *Consect.* 2. *Definit.* 28. and by further subtracting from each side bc, you'l have $ac-bc=ec$, that is, to the Product of $a-b$ by c. Q. E D.

CONSECTARY.

Since $ac-bc$ is the Product of $a-b$ by c, it is manifest also, that if $ac-bc$ be divided by c, you'l have $a-b$ for the Quotient; and so always a Positive Quantity (as ac) divided by a Positive one, c, will give a Positive Quotient; but a Negative Quantity $-bc$ divided by a Positive one, will give a Negative Quotient.

Proposition V.

If a Negative Quantity be multiplied by a Negative one, the Product will be Positive.

Exposition.

Suppose $a-b$ is to be multiplied by $-c$; it is certain, that a multiplied by $-c$ will give the Negative Quantity $-ac$, by *Prop.* 4. but $-b$ multiplied by the same $-c$ will produce $+bc$, and so the whole Product will be $-ac+bc$.

A Demonstration like the former.

Suppose $a-b=e$, then will $-ec=$ the Product of $a-b$ by $-c$: and since $a-b$ is $=e$, adding b on both sides you'l have $a=e+b$, by *Schol. Definit.* 26. and multiplying both sides by $-c$, you'l have $ac=-ec-bc$, by *Prop.* 4. and *Consect.* 2. *Definit.* 28. and by adding bc on both sides, you'l have

$-ac$

$-ac+bc=-ec$, i. e. to the Product of $a-b$ by $-c$ Q. E. D.

CONSECTARYS.

I. Since therefore $a-c+bc$ arises from $a-b$ by $-c$, it is manifest, that if $-ac+bc$ be divided again by $-c$, you will again have $a-b$, and consequently a Negative Quantity divided by Negative, will give a Positive Quotient but a Positive Quantity $+bc$ divided by a Negative one, will give a negative Quotient $-b$.

II. We have therefore the Foundation and Demonstration of the Rules of Specious Computation, in the multiplication and division of Compounded Quantities, *viz.* that the same Signs multiplied together (as $+$ by $+$ or $-$ by $-$) give $+$; but different (as $+$ by $-$ or $-$ by $+$) give the Sign $-$. Which Rules the following Examples will Illustrate, as also several other we shall meet with in the following Chapter.

Multiplication.

$$\begin{array}{l} a+b \\ a-b \\ \hline aa+ab \\ \quad\;\; -ab-bb \\ \hline aa-bb \end{array} \qquad\qquad \begin{array}{l} a-b \\ c-d \\ \hline ac-cb \\ \quad\;\; -ad+bd \\ \hline ac-cb-ad+bd \end{array}$$

$$\begin{array}{l} 3d-4e+2f \\ 5d-g \\ \hline 15dd-20de-10df \\ \qquad\quad -3dg+4ge-2gf \\ \hline 15dd-20de+10df-3dg+4ge-2gf \end{array}$$

Division.

$$\text{by}\ \frac{aa-bb}{a+b}\left(a-b\right. \quad\Big|\quad \text{by}\ \frac{ac-ad-bc+bd}{c+d\quad\;\; c-d}\left(a-b\right.$$

CHAP. I

CAAP. II.

Of the Powers of QUANTITIES.

Containing (after a compendious Way) most part of the 2d Book of *Euclid*; and the Appendix of *Clavius* to *Lib.* 9. *Prop.* 14.

Proposition VI.

IF any whole Quantity be divided into two parts (α) the Rectangle contained under the whole, and one of its parts, is equal to the Square of the same part, and the Rectangle contain'd under both the parts.

(α) *Eucl. lib.* 2. *prop.* 3. *A. C. also the third.*

Demonstration.

Let $a + b$ represent the whole	$a + b$
b one part of it, or	a the other.
$ab + bb$ the Rectangle,	$aa + ab$ the Rectangle.
(See *Fig.* 50.)	Q. E. D.

Proposition VII.

IF a whole Quantity be divided into two parts (β) the Square of the whole is equal to the Squares of both those parts and 2 Rectangles contained under them.

(β) *Eucl. Prop.* 4. *lib.* 2.

Demonstration.

This is evident from the preceding, and may moreover thus appear further.

Let

Let the Parts be a and b, then will the whole be $a+b$
Which if you multiply by it self $a+b$

$$\begin{array}{r} aa+ab \\ ab+bb \\ \hline aa+2ab+bb \end{array}$$

You have the Square $aa+2ab+bb$
(See *Fig.* 51. N. 1.) Q. E. D.

CONSECTARYS.

I. HEnce you have the Original Rule for Extracting of Square Roots, as we have shewn after *Definition* 30. and here have further Illustrated in *Scheme* N° 2.

II. Hence it naturally follows, that the Square of double any Side is Quadruple of the Square of that Side taken singly.

III. Hence also you have the addition of surd Numbers, or in general of surd Quantities, by help of the following Rule (supposing in the mean while their Multiplication:) Suppose these 2 Surds $\sqrt{8}$ and $\sqrt{18}$, or more generally $\sqrt{75aa}$ and $\sqrt{27aa}$, are to be added together; first add their Squares 8 and 18, &c. then double their Rectangle ($\sqrt{144}$) that is, multiply it by the $\sqrt{4}$, and then the double of this $\sqrt{576}$, *i. e.* having extracted the Square Root, (24) and added it to the Sum of the first Squares (26) the Root of the whole Summ (50) *viz.* $\sqrt{50}$, is the Sum of the two surd Quantities first proposed.

SCHOLIUM.

BUT if it happens that the Root of the double Product cannot be expressed by a Rational Number (as, when the proposed Quantities are Surds, as $\sqrt{3}$ and $\sqrt{7}$, to whose Squares $3+7$, *i. e.* 10, you must add the double Product of $\sqrt{7}$ by $\sqrt{3}$, *i. e.* $\sqrt{84}$, which cannot be expressed by a Rational Number) then that double Product must be joined under a Surd Form, or Radical Sign, to the Sum of the Squares (thus, *viz.* $10+\sqrt{84}$) and to this whole Aggregate prefix another Radical Sign, thus, $\sqrt{10+\sqrt{84}}$; or also you may only simply join the Surd Quantities proposed by the Sign $+$ thus, $\sqrt{3}+\sqrt{7}$. Here also you may note, that the two *Surd Quantities* proposed in

in the first case of *Consectary* 2. are called *Communicants*; in the other case of this *Scholium*, *Non-Communicants*: For in this case each quantity under the Radical Sign may be divided by some Square, and have the same Quotient (*e. g.* 8 and 18, may be divided the first by 4, the other by 9, and the Quotient of both will be 2; likewise $75aa$ and $27aa$ may be divided, the one by $25aa$, the other by $9aa$, the Quotient of both being 3; and then if the Quotient on both sides be left under the Radical Sign, and the Root of the dividing Square set before it, the same quantities will be rightly expressed under this form: $2\sqrt{2}$ and $3\sqrt{2}$, also $5a\sqrt{3}$ and $3a\sqrt{3}$; and then the addition is easie, *viz.* only collecting or adding together the Quantities prefixt to the Radical Sign; so that the Sums will be of the one $5\sqrt{2}$ and of the other $8a\sqrt{3}$, which are indeed the same we have shewn in *Consect.* 2. For if contrarywise we square the Quantities that stand without, or are prefixt to the Radical Sign, and then set those Squares (25 and $64aa$) under the Radical Sign, multiplying by the Number prefixt to it, you'l have for the one $\sqrt{50}$, for the other $\sqrt{192aa}$ (*Consect.* after *Schol- Prop.* 22.)

Proposition VIII.

If any whole Quantity (*viz.* Line or Number) be divided (*a*) into two equal parts, and two unequal ones, the Rectangle of the unequal ones, together with the Square of (the intermediate part or) the difference of the equal part from the unequal one, is equal the Square of the half.

(*a*) *Eucl.* & *Clav.* 5.

An Universal **Demonstration.**

Suppose the parts to be a and a, and the whole $2a$; let one of the unequal Parts be b, the other will be $2a-b$, and the difference between the equal and unequal part $a-b$.

The

$$\begin{array}{lrlr} \text{The equal ones} & 2a-b & \text{Difference} & a-b \\ & +b & & a-b \\ \hline \text{Rectangle} & 2ab-bb & & aa-ab \\ & & & -ab+bb \\ \hline & & & aa-2ab+bb\square \end{array}$$

The Sum will be aa (the other parts destroying one another) Q. E. D. (*Vid. Fig.* 52.)

Proposition IX.

IF to any whole Quantity divided into two equal parts (*a*) you add another Quantity of the same kind, the Rectangle or Product made of the whole and the part added, multiplied by that part added, together with the square of the half, will be equal to the Square of the Quantity compounded of that half, and the part added.

(*a*) *Eucl.* & *Clav.* 6.

Demonstration.

Let the whole be called $2a$, the part added b, then the quantity compounded of the whole and the part added will be $2a+b$; and that compounded of the half and the part added $a+b$.

The Quantity compounded of the whole, and the part added, is,

$$\begin{array}{lrrr} & 2a+b & \text{the half } a & \text{Comp. } a+b \\ \text{Multip. by the part added} & b & a & a+b \\ \hline & & & aa+ab \\ & & & ab+bb \\ \hline & 2ab+bb\square & \text{of the half } aa & =\square aa+2ab+bb \end{array}$$

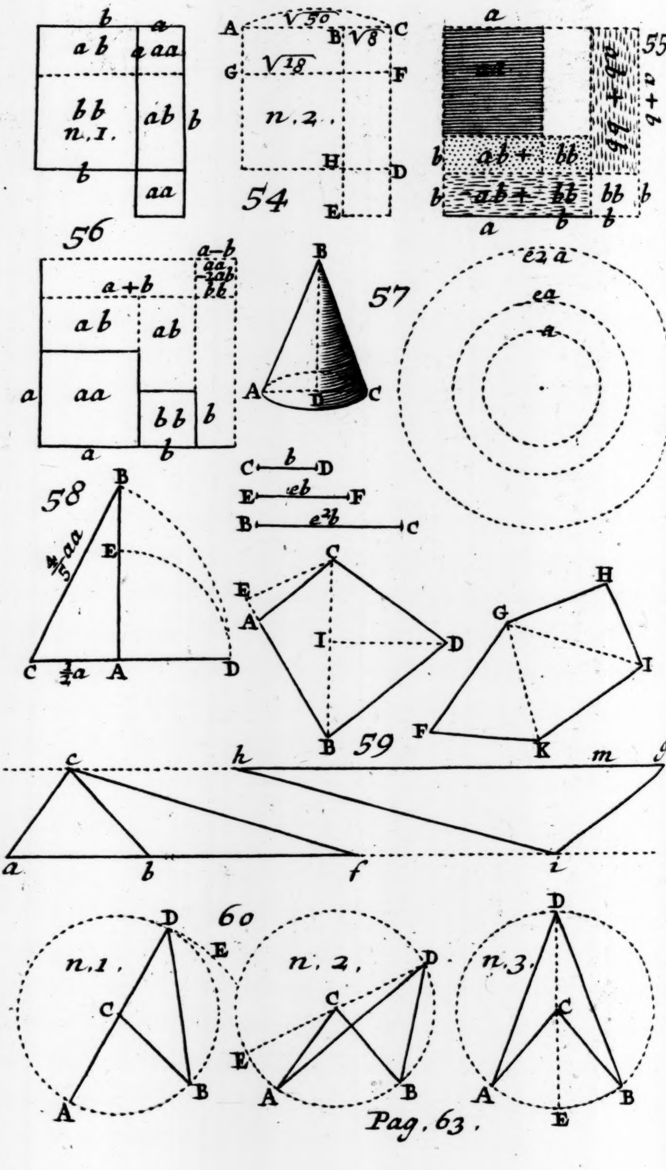
(*Vid. Fig.* 53.) Q. E. D.

Propositiou X.

IF a Quantity be divided any how into (*b*) two parts, the Square of the whole, together with that of one of its parts, is equal to two Rectangles contained under the whole and the first part, together with the Square of the other part.

(*b*) *Eucl.* & *Clav.prop.*7.*l.*2

The



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The Universal Demonstration.

Let a be one part and b the other, the whole $a+b$

$a+b$ the whole. The whole $a+b$

a the first part * $a+b$

$aa+ab$ $aa+ab$

2 $ab+bb$

$2aa+2ab$ the double rectangle □ of the whole $aa+2ab+bb$

add bb the □ of the other part * add aa

Sum $2aa+2ab+bb$ = to the Sum . . . . $2aa+2ab+bb$

(*Vid. Fig.* 54. N° 1.) Q.E.D.

CONSECTARY.

HEnce you have the Subtraction of Surd Numbers, or more generally of Surd Quantities, by help of the following Rule.

Add the Squares of the given Roots according to Consect. 3. Prop. 7. *and from their Sum subtract the double Rectangle of their Roots; the Root of the Remainder will be the difference sought of the given Quantities.*

As, if the $\surd 8$ (BC) is to be subtracted from $\surd 50$ AC (*Fig.* 54, N° 1.) you must add 50, *i. e.* the whole Square AD) and 8,(*i. e.* the other Square superadded DE,) and the Sum will be 58, equal to the two Rectangles AF and FE + □ GH, by this Prop. I find therefore those two Rectangles by multiplying $\surd 50$ by $\surd 8$, and then the Product $\surd 400$ by 2 or $\surd 4$, thereby to obtain the double Rectangle $\surd 1600$, *i. e.* (having actually Extracted the Root) 40. This double Rectangle therefore or 40, being subtracted out of the superiour Sum, the remainder 18 will be the □GH, and so its Root (*viz.* $\surd 18$) gives the required Difference between the given Surd Quantities.

SCHOLION.

BUT this Subtraction may be performed yet a shorter way, if each quantity under its Radical can be divided by some square,

square, so that the same Quotient may come out on both sides that is, if the Surd Quantities are *Communicants*, as *e. g.* $\sqrt{5}$ (the number 50 being divided by 25) is equal to $5\sqrt{2}$ and $\sqrt{8}$ to $2\sqrt{2}$; for then the numbers prefixt to the Radical Sign being subtracted from one another (*viz.* $2\sqrt{2}$ from $5\sqrt{2}$) you have immediately the remainder or difference $3\sqrt{2}$, *i. e.* $\sqrt{1}$ But if the proposed Quantities are not Communicants (as if the $\sqrt{3}$ is to be subtracted from $\sqrt{7}$) the remainder may be briefly expressed by means of the Sign — thus, $\sqrt{7}-\sqrt{3}$, or according to the foregoing *Consectary*, thus, $\sqrt{10-\sqrt{84}}$.

Proposition XI.

If any *Quantity be divided into two parts*, (*a*) *the Quadruple Rectangle contained under the whole and one of its parts, together with the Square of its other parts, will be equal to the Square of the Quantity compounded of the whole and the other part.*

Demonstration.

Suppose $a+b$ the whole.
b one part.

$ab+bb$ the Rectangle of these two.
mult. by 4

$4ab+4bb$ the Quadruple Rectangle.
Add aa the Square of the other part.

Sum $aa+4ab+bb$

The Quantity compounded of the whole and the first part $\left.\begin{matrix} a+2b \\ a+2b \end{matrix}\right.$

$aa+2ba$
$2ba+4bb$

$aa+4ba+4bb$

Square of the Compound Quantity Q. E.

(*Vid. Fig.* 55.)

(*a*) *Eucl.* & *Clav. prop.* 8.

Proposition

Proposition XII.

IF any Quantity be divided into two equal parts (β) and into two other unequal ones, the Squares of the unequal parts taken together will be double the Square of half the quantity, and the Square of the difference, *viz.* of the equal and unequal part, taken together.

(β) *Eucl* & *Clav. prop.* 9.

Demonstration.

Suppose the equal parts to be a and a, the difference (b) the greater of the unequal Parts to be $a+b$, the less $a-b$.

The greater part	The less	Half	Difference
$a+b$	$a-b$	a	b
$a+b$	$a-b$	a	b
$\square\, aa+2ab+bb$	$aa-2ab+bb$	$\square\, aa$	$\square\, bb$

Sum of these $2aa+2abb$ | Sum $aa+bb$

Q. E. D. (*Vid. Fig.* 56.)

Proposition XIII.

IF to any whole Quantity (α) divided into two equal parts there be added another Quantity of the same kind, the Square of the Quantity compounded of the whole, and the quantity added, together with the square of the quantity added, will be double the square of the half the quantity, and the square of the Sum of the half and the part added taken together.

(α) *Eucl.* & *Clav.* x.

Demonstration.

Suppose the whole to be $2a$, the half parts a and a, the quantity added b; then the quantity compounded of the whole and the quantity added, will be $2a+b$, and that of the half and the quantity added $a+b$.

Comp. of the whole and quantity added	Half	Qu. compouded of hal[f] and qu. added,
$2a+b$	a	$a+$
$2a+b$	a	$a+$
$4aa+2ab$		
$2ab+bb$	$\square\ aa$	$\square aa+2ab+$
$\square\ 4aa+4ab+bb$		Sum $2aa+2ab+$
$\square$ Qu. added, bb		Manifestly the half of the former Sum. Q.E.D.
Sum $4aa+4ab+2bb$		

CHAP. III.

Of Progression, or Arithmetical Proportionals.

Proposition XIV.

IF there are 3 Quantities in continued Progression, or Arithmetical continued Proportion, the Sum of the Extrem[es] is double of the middle Term.

Demonstration.

Such are *e.g.* a, $a+x$, $a+2x$ ascending,
or a, $a-x$, $a-2x$ descending.

By *Definition* 32. the Sum of the Extremes in the first $2a+2x$, in the latter $2a-2x$; in both manifestly double the middle Term Q.E.D.

Proposition XV.

IF there are 4 of these Continued Proportionals, the Sum [of] the Extreme Terms is equal to the Sum of the me[an] Terms.

Dem[onstration.]

Demonstration.

Such are *e. g.* a, $a+x$, $a+2x$, $a+3x$, &c. aſcending, or a, $a-x$, $a-2x$, $a-3x$, &c deſcending; in the one the Sum of the Extremes is $2a+3x$, in the other $2a-3x$; and alſo of the means $2a+3x$ and $2a-3x$ Q. E D.

Propoſition XVI.

IF there are never ſo many of theſe continued Proportionals, the Sum of the Extremes is always equal to the Sum of any 2 other of them, equally remote from the Extremes, or alſo double of the middle Term, if the number of the Terms is odd.

Demonſtration.

Suppoſe a, $a+x$, $a+2x$, $a+3x$, $a+4x$, $a+5x$, $a+6x$, *&c.* or a, $a-x$, $a-2x$, $a-3x$, $a-4x$, $a-5x$, $a-6x$, and the Sum of the Extremes, as alſo of any 2 equally remote from the Extremes, and the double of the middle Term is in the firſt Series $2a+6x$, in the latter $2a-6x$, *&c.* Q.E.D.

SCHOLIUM I.

NOR can we doubt but that this will always be ſo, how far ſoever the Progreſſion be continued; if you conſider that the laſt Term contains in it ſelf the firſt, and moreover the difference ſo many times taken, as is the number of Terms excepting one, but that the firſt has no difference added to it; and therefore tho the laſt ſince one contains one difference leſs than the laſt; the ſecond on the contrary has one more than the firſt, and conſequently the Sum of the one will neceſſarily be equal to the Sum of the other; and in like manner the laſt except two, contains two Differences leſs than the laſt; but, on the contrary, the third exceeds the firſt by a double Difference, the double Difference being added to it, *&c.* as is obvious to the Eye in our firſt univerſal Example. Hence you have theſe

CONSECTARYS.

I. YOU may obtain the Sum of any Terms in Arithmetical Proportion, if the Sum of the Extremes be multiplied by half the number of Terms, or (which is the same thing) half the Sum by the number of Terms,

II. To obtain therefore the Sum of 600, or never so many such Terms, you need only have the Extremes and the number of Terms: So that you have a very compendious Way of proceeding in Questions that are solvible by these Progressions, if, having the first Term and Difference of the Progression given, you can obtain the last, neglecting the intermediate ones.

III. But you may obtain the last Term, by Multiplying the given given Difference by the given Number of Terms lessened by Unity, and then adding the first Term to the Product; as is evident from the preceding *Scholium*.

IV. Hence we may easily deduce this Theorem, that the Sum of any Arithmetical Progression beginning from o, is subduple of the Sum of so many Terms, equal to the greatest, as is the number of Terms of that Progression. For if the first Term is o and the last x, and the given number of Terms a, the Sum of the Progression will be $\frac{1}{2}ax$, by *Consect.* 1. but the Sum of so many Terms equal to the greatest, ax. Q.E.D.

SCHOLIUM II.

NOw if any one would be satisfied of the truth of this last *Consect.* without the literal or specious Notes, let him consider, that if the first Term be supposed to be o, the last (whatever it is) will be the sum of the Extremes. The last therefore multiplied by half the number of Terms, gives the Sum of the Progression, by *Consect.* 1. and the same last Term multiplied by the whole number of Terms, gives the Sum of so many Terms equal to the greatest. But that this must needs be double of the precedent 'tis evident, because any Multiplicand being multiplied by a double multiplier, must needs give a double Product. Now as this Consectary will be of singular Use to us hereafter for Demonstrating several Propositions, so the three former are the very same Practical Rules of Arithmetick, which are commonly made use

uſe of in Arithmetical Progreſſions; for the Illuſtration whereof *Swenterus* gives us ſeveral Ingenious Examples in his *Delic.* part I. Queſt. 70. *&c.*

CHAP. IV.

Of Geometrical Proportion in General.

Propoſition XVII.

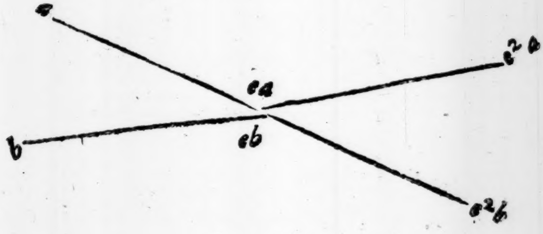
IF there are three Quantities continually (*a*) Proportional, the Rectangle of the Extremes, is equal to the Square of the mean Term.

Demonſtration.

Such are *e. g.*	$a,\ ea,\ e^2a,$		The mean Term,	ea
The Extremes	e^2a			ea
	a			
Rectangle	e^2a^2		Square	e^2a^2 Q.E.D.

SCHOLIUM.

MOreover if three Quantities on each ſide are in the ſame Continual Proportion, as



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the Rectangles of the Extremes made Cross-ways, are equal to the Rectangle of the mean Term; being every way e^2ab.

Whence by the way may appear that Proposition of *Archimedes*: (β) *That the Surface of a Right Cone is equal to the Circle, whose Radius is a mean Proportional between the Side of that Cone and the Semidiameter of the Base.* For suppose EF to be a mean Proportional between the side of the Cone BC (*Fig.* 57.) and the Semidiameter of the Base CD, since an equal number of Peripherys answer to an equal number of Radii in the same Proportion; half the Product of the first Line BC into the last Periphery, $\frac{1}{2}e^2ab$ (that is, by *Consect.* 4. *Definit.* 18. the Surface of the given Cone) will be equal to half Product of the mean Line into the mean Periphery, $\frac{1}{2}e^2ab$ (*i. e.* by *Consect.* 2. *Definit.* 15.) to the Area of the Circle of the mean Proportional EF. Q. E. D.

(α) Eucl. 17 l. 6 & 20 l. 7.
(β) lib. 1. de Sphær. & Cyl. Prop. 14.

The same Proposition of *Archimedes* may also be Demonstrated after this Way: If the side of the Cone BC be called b, and the Semidiameter of the Base a)so that the Periphery may, by *Consect.* 1. *Definit.* 31. be $2ea$, and so the Surface of the Cone, by *Consect.* 4. *Definit.* 18. eab) the $\sqrt{ab}$ will be a mean proportional between b and a, by this 17th *Proposition*; which being taken for *Radius*, the whole Diameter will be $\sqrt{2ab}$, and the Periphery $2e\sqrt{ab}$; therefore by *Consect.* 2. *Definit.* 15. half the Radius $\frac{1}{2}\sqrt{ab}$ multiplied by the Periphery (since $\sqrt{ab}$ multiplied by $\sqrt{ab}$ necessarily produces ab) will give you the Area of the Circle by that mean (*a*) Proportional, equal to the Surface of the given Cone, which before was expressed in the same Terms. Q. E. D.

Hence also naturally flows this other Proposition, That the Surface of the Cone ($\frac{1}{2}e^2ab$) is to its Base ($\frac{1}{2}ab$) as the Side of the Cone (e^2b) is to the Radius of the Base b, as may appear from the Terms.

Proposition XVIII.

IF (*b*) 4 Quantities are Proportional, either continuedly or dircretely, the Product of the Extremes is equal to the Product of the Means.

(*a*) lib. 1. de Sphær. & Cyl. Prop. 15.
(*b*) Eucl. 16 l. 6 & 19 l. 7.

Demon-

Demonstration.

Suppose one Continual Proportional, a, ea, e^2a, e^3a.

Extremes e^3a | Means e^2a
a | ea

Prod. e^3aa = Prod. e^3aa. Q.E.D.

SCHOLIUM.

ON this Theorem is founded the *Rule of Three* in Arithmetick; so called because having 3 Numbers, (2. 5. 8.) it finds an unknown fourth Proportional. For altho this fourth be, as we have said, unknown, yets its Product by 2 is known, because the same with the Product of the Means, 5 and 8. Wherefore the Rule directs to multiply the third by the second, that you may thereby obtain the Product of the Extremes: which divided by one of the Extremes, *viz.* the first, necessarily gives the other, *i. e.* the fourth sought.

Proposition XIX.

IF 2 Products (on the other side) arising from the Multiplication of 2 Quantities, are equal, those 4 Quantities will be at least directly Proportional.

Demonstration.

Suppose eba be the equal Product of the Extremes, and eab of the Means; the Extremes will either be eb and a, or e and ba, or b and ea, as also the Means. But what way soever either is taken, there can be no other Disposition or placing of them, than one of the following.

I eb eb a a
—— ee ab
—— ea b; or inversly.
—— a eb
—— ab e

——— *b* *ea*
2 *e* *e* *ba* *ba*
——— *eb* *a*
——— *ea* *b*; or inversly.
——— *ba* *e*
——— *a* *eb*
——— *b* *ea*
3 *b* *b* *ea* *ea*
——— *a* *eb*
——— *ba* *e*; or inversly.
——— *ea* *b*
——— *eb* *a*
——— *e* *ba*; or inverting the Order of them all.

In all these Dispositions there may be immediately seen a Geometrical Proportion, by what we have in *Definition* 31 and 33.

CONSECTARSY.

I. As we have shewn one *S*ign of Proportionality in the Definition of it, *viz*. That the same Quotient will arise by dividing the Consequents by the Antecedents; so now we have another *S*ign of it, *viz*. The Equality of the Products of the Extremes and Means.

II. By a bare *S*ubsumption may hence appear the Truth of *Prop*. 14. *lib*. 6. *Euclid*. at least partly: Which we shall yet more commodiously shew hereafter.

Proposition XX.

If there are never so many Continual Proportionals, the Product of the Extremes is equal to the Product of any 2 of the Means that are equally distant from the Extremes, as also to the *S*quare of the mean or middle Term, if the Terms are odd.

Demonstration.

*S*uch are *e. g.* a, ea, e^2a, e^3a, e^4a, e^5a, e^6a, &c. and the Product of the Extremes, and of any two Terms equally remote from

them, and the Square of the mean or middle Term, every where e^6aa. Q. E. D.

SCHOLIUM I.

NOR can there be any doubt but this will always be so, how far soever the Progression is continued ; if [illegible] con- sider that the last Term always contains the first, [illegible] way of Reason, so many times multiplied as is the place of that Term in the rank of Terms, excepting one. Altho therefore the last Term but one is in one degree of its Reason less than [illegible] last, the second on the contrary, is in one more than [illegible] re- fore the Product of the one will nec ssari[illegible] e Product of the other. Thus also the la[illegible] Degrees of Proportion lower than the [illegible] being to be multiplied into that, exc[illegible] of the Proportion, &c. as may be seen in [illegible] Universal Ex[illegible] Hence you have the following

CONSECTARIES.

I. HAving some of the Terms given in a Continual Pro- portion (*e. g.* suppose 10) you may easily find any other that shall be required (*e. g.* the 17th) as the last; If the 2 Terms given, being equally remote from the first and that required (as are *e. g.* the eigth and tenth) be multiplied by one another, and this Product, like that also of the Extremes, be divided by the first.

II. But this may be performed easier, if you moreover take in this Observation, That if, *e. g.* never so many places of pro- portionals, passing over the the first, be noted or marked by Ordinals or Numbers according to their places (as in this uni- versal Example)

$$a,\ ea,\ e^2a,\ e^3a,\ e^4a,\ e^5a,\ e^6a,$$
$$\text{I. II. III. IV. V. VI.}$$

The place of the 7th Term is (*e. g.*) VI. (and so the place of any other of them being less by Unity than its number is among the Terms) and also composed of the places of any other equal- ly distant from the Extremes, *e. g.* V. and I. IV. and II. or twice III. &c.

III. Here

III. Here you have the Foundation of the Logarithms, *i. e.* of a Compendious Way of Arithmetick, never enough to be praised. For if, *e. g.* a rank of Numbers from Unity, continually Proportional, be signed or noted with their Ordinals, as we have said, as Logarithms,

1.	2.	4.	8.	16.	32.	64.	128.	256,	&c.
	I.	II.	III.	IV.	V.	VI.	VII.	VIII.	

and any two of them (as 8 and 32) are to be multiplied together; add their Logarithms III and V, and their Sum VIII gives you the Logarithm of their Product 256, as the Term equally remote from the 2 given ones and the first, and whose Product with the first (which is Unity) *i. e.* it self will be equal to the Product of the Numbers to be multiplied: And contrariwise, if, *e. g.* 128 is to be divided by 4, subtracting the Logarithm of the first II from the Logarithm of the second VII the remaining Logarithm V points out the number sought 32: so that after this way the Multiplication of Proportionals is by a wonderful Compendium, turned into Addition, and the Division into Subtraction, and Extraction of the Square Root into Bisecting or Halving, (for the Logarithm of the Square Number 16 being Bisected, the half II gives the Root sought 4) of the Cube Root into Trisection (for the Logarithm of the Cube 64 being Trisected, the third part gives the Cubick Root sought 4).

SCHOLIUM II.

THat we may exhibit the whole Reason of this admirable Artifice (which about 35 years ago was found out by the Honourable Lord *John Naper* Baron of *Merchiston* in *Scotland*, and published something difficult, but afterwards render'd much easier and brought to perfection by *Henry Briggs*, the first *Savilian* Professor of Geometry at *Oxford.*) I say that we may exhibit the whole Reason of it in a Synopsis, after an easie way; when its use appear'd so very Considerable in the great Numbers in the Tables of Sines and Tangents, nor yet could they be useful without mixing vulgar Numbers with them, especially in the Practical Parts of Geometry, the business was to ac-

commoda-

commodate this Logarithmical Artifice to them both. First therefore that Artists might assign Logarithms to all the common Numbers proceeding from 1 to 1000 and 10000, *&c.* they first of all pick out those which proceed in continued Geometrical Proportion, and particularly, tho arbitrariously, those which increase in a Decuple Proportion, *e. g.* 1. 10 100. 1000. 10000, *&c.*

But now to fit them according to the Foundation of *Consect.* 3. a Series of Ordinals in Arithmetical Progression, we do'nt only substitute the simple Number 1, 2, 3, *&c.* but augmented with several Cyphers after them, that so we may also assign their Logarithms in whole Numbers to the intermediate Numbers between 1 and 10, 10 and 100, *&c.* Wherefore, by this first Supposition, Logarithms in Arithmetical Proportion, answer to those Numbers in Geometrical Proportion, after the way we here see,

	1	10	100
Log.	0000000	10000000	20000000
		1000	10000
		30000000	40000000, *&c.*

As that they also exhibit certain Characteristical initial Notes, whereby you may see, that all the Logarithms between 1 and 10 begin from 0, the rest between 10 and 100 from 1, the next from 100 to 1000 from 2, *&c.*

The Logarithms of the Primary Proportional Numbers being thus found, there remain'd the Logarithms of the intermediate Numbers between these to be found: For the making of which, after different ways, several Rules might be given drawn from the Nature of Logarithms, and already shewn in *Consect.* 3. See *Briggs*'s *Arithmetica Logarithmica*, and *Gellibrand*'s *Trigonometria Britannica*; the first whereof, *chap.* 5. and the following, shews at length both ways delivered by *Neper* in his *Appendix.* But the business is done more simply by *A. Vlacq.* in his Tables of Sines *&c.* whose mind we will yet further explain thus: If you are to find, *e. g.* the Logarithm of the Number 9, between 1 and 10, augmented by as many Cyphers as you added to the Logarithm of 10, or the rest of the Proportionals (*h. e.* between

10000000

10000000 and 100000000) you must find a Geometri Mean Proportional, *viz.* by multiplying these Numbers togeth and extracting the Square Root out of the Product, by Pr 17. Now if this Mean Proportional be less than 9 augment by as many Cyphers, between it and the former Denary Nu ber you must find a second mean Proportional, then betw this and that same a third; and so a fourth, &c. but if it greater, then you must find a mean Proportional between and the next less, &c. till at length after several Operatio you obtain the number 9999998, approaching near 90000000. Now if between the Logarithm of Unity a Ten (*i. e.* between 0 and 10000000) you take an Arithm tical Mean Proportional (05000000) by Bisecting their S by Prop. 14. and then between this and the same Logarithm Ten, you take another mean, and so a third and a fourth, & at length you will obtain that which answers to the last abo mentioned, *viz.* 9. See the following Specimen.

A TABLE of the Geometrical Proportionals betwe 1 and 10, augmented by 7 Cyphers, and of t Arithmetical Proportionals between 0 *and* 1000000 *being the Logarithms corresponding to them.*

Geometrical Mean Proportionals.		*Arithmetical Logar. mean Proportionals*
31622777	First,	05000000
56234132	Second,	07500000
74989426	Third,	08750000
86596435	Fourth,	09375000
93057205	Fifth,	09687500
89768698	Sixth,	09531250
91398327	Seventh,	39609375
90579847	Eighth,	09570312
90173360	Ninth,	09550781
89970801	Tenth.	09541015

Whi

Which is thus made: In the first Table a Geometrical Mean roportional between 10000 000 and 100 000 000 the first Number of it; then another Mean between that and e same last 100000000, gives the second; and so to the th, 93057205. Which, since it is already greater than the ovenary, another Mean between it and the precedent fourth, comes in order a sixth, but sensibly less than the Novenary. herefore between it and the fifth you will have a seventh ean yet greater than the Novenary; and between the sixth d seventh, an eighth, somwhat nearer to the Novenary, but t yet sensibly equal, but somewhat bigger; moreover between e sixth and eighth you will have a ninth, between the ninth d sixth a tenth gradually approaching nearer the Novenary, but t somewhat sensibly differing from it. Now if you conue this inquiry of a mean Proportional between this tenth, somewhat too little, and the precedent ninth as somewhat o big, and so onwards, you will at length obtain the Numer 8999 9998, only differing two in the last place from the ovenary Number augmented by seven Cyphers, and conseently insensibly from the Novenary it self. But for the Logarithm of this in the second Column, by the same process you e to find Arithmetical Mean Proportionals between every 2 ogarithms answering to every two of the superiour ones, till you nd, *e. g.* the Logarithm of the tenth Number 09541015, nd so at length the Logarithm of the last, not sensibly differng from the Novenary, 09542425.

Thus having found, with a great deal of labour, but also ith a great deal of advantage to those that make use of them, e Logarithms of some of the numbers between 1 and 10, and o and 100, *&c.* you may find innumerable ones of the other termediate Numbers with much less labour, *viz.* by the help f some Rules, which may be thus obtain'd from *Consect.* 3 of e precedent Proposition. *The Sum of the Logarithms of the umber Multiplying and the Multiplicand, gives the Logarithm of the roduct.* 2. *The Logarithm of the Divisor subtracted from the Logarithm of the Dividend, leaves the Logarithm of the Quotient*: *e Logarithm of any number doubled, is the Logarithm of the Square, ripled of the Cube,* &c. 4 *The half Logarithm of any number is e Logarithm of the Square Root of that number, the third part of of the Cube Root, &c.* Thus, *e. g.* if you have found the Logarithm

garithm of the number 9, after the way we have ſhewn, by the ſame reaſon you may find the Logarithm of the number 5 (or) by finding mean Proportionals between the ſecond and the firſt number of our Table, and between their Logarithms, &c. and by means of theſe 2 Logarithms you may obtain ſeveral others: Firſt, ſince 10 divided by 5 gives 2; if the Logarithm of 5 be ſubtracted from the Logarithm of 10, you'l have the Logarithm of 2, by Rule the ſecond. Secondly, ſince 10 multiplied by 2 makes 20, and by 9 makes 90, by adding the Logarithms of 10 and 2, and 10 and 9, you'l have the Logarithms of the numbers 90 and 20, by Rule 1. Thirdly, Since 9 is a Square, and its Root 3, half the Logarithm of 9 gives the Logarithm of 3, by Rule 4. ſince 90 divided by 3 gives 30, the Logarithm of this number may be had by ſubtracting the Logarithm of 3 from the Logarithm of 90, by Rule the ſecond. Fifthly, 5 and 9 ſquared make 25 and 81, the Logarithms of 5 and 9 doubled, give the Logarithms of theſe numbers, by Rule 3. In like manner, ſixthly, the Sum of the Logarithms of 2 and 3, or the Difference of the Logarithms of 5 and 30, give the Logarithm of 6, and the Sum of the Logarithms of 3 and 6, or 2 and 9, gives the Logarithm of 18; the Logarithm of 6 doubled, gives the Logarithm of 36, &c. And after this way you may find and reduce into Tables, the Logarithms of Vulgar Numbers from 1 to 1000 (as in the Tables of *Strauch*. p. 182, and the following) or to 100000 (as in the *Chiliads* of *Briggs*) But as to the manner of deducing the Tables of Sines and Tangents from theſe Logarithms of Vulgar Numbers, we will ſhew it in *Schol.* of *Prop.* 55, only hinting this one thing before-hand; that this Artifice of making Logarithms is elegantly ſet forth by *Pardies* in his *Elements of Geometry*, pt 112. by a certain Curve Line thence called the *Logarithmical Line*; by the help whereof he ſuppoſes Logarithms may be eaſily made; and having found thoſe of the numbers between 1000 and 10000, he ſhews, that all others may be eaſily had between 1 and 1000. Wherefore we ſhall Diſcourſe more largely in *Schol. Definit.* 15. *lib.* 2.

Propoſition

Proposition XXI.

I*F the first Term of never so many Continual Proportionals, be subtracted from the last, and the Remainder divided by the name of the Reason or Proportion lessen'd by Unity, the Quotient will be equal to the Sum of all except the last.*

Demonstration.

ea

e^2a

e^3a

e^4a

e^5a

The last Term less the first $e^6a - a$

Divided by the name of the Reason lessen'd by unity.	$e-1$ $e-1$ $e-1$ $e-1$ $e-1$ $e-1$ $e-1$	* *Quote.* $e^5a + e^4a + e^3a + e^2a + ea + a$; And it is evident from the Operation, that the same will always happen tho the number of Terms be continued never so far.

CONSECTARYS.

I. WHerefore in adding never so great a Series of Geometrical Proportionals, since it is enough that the first and last Term, and the Name of the Reason be known, *by this Prop.* and having found at least some of the Terms of the Proportion, any other may be afterwards found, whose place will be compounded of the places of the two Antecedent ones, according to *Consect.* 2. *Prop.* 20. *viz.* by Multiplying the Terms answering to the two above-mentioned places, and dividing the Product by the first Term; thence it will be very easie to add a great Series of Proportionals into one Sum, tho the particular separate Terms remain almost all of them unknown.

SCHOLIUM

SCHOLIUM.

THese are the same Practical Arithmetical Rules concernin Geometrical Progressions; for the illustration of whic *Swenterus in Delic.* has given us so many pleasant Examples, *li* 1. *Prop.* 59. and *fol.* First of all, that famous Example is of th kind which relates to the Chequer-work'd Table or Board t fling Dice on, with its 64 little Squares, which Dr. *Wa* has translated out of the Arabick of *Ebn Chalecan*, into Latin in *Oper. Mathem. part.* 1. *Chap.* 31. for the illustration of which we have heretofore composed an Exercitation, and shall her only note these few things: If there are supposed 64 Terms o double Proportion from Unity, and the first of them, not with their local Numbers, are these that follow;

1	2	4	8	16	32	64	128
	I	II	III	IV	V	VI	VII

You may have the Term of the 13th place, 8192, by mu tiplying together the VIth and VIIth place; and the Ter of the XXVIth place, by squaring or multiplying this new Pro duct again by it self, and moreover the Term of the LI place, by multiplying that Product again by itself; and furthe more the Term of the LIXth place, by multiplication of the num ber last found by the number of the VIIth place, and lastly the Term of the LXIIId place (*i. e.* the last in the proposed Se ries) by multiplying this last of all by the number of the IV place.

II. Moreover you may, by this Art, collect infinite Seri of Proportional Terms into one Sum, altho it is impossible t run over all the Terms separately, because infinite. *e.g.* in continued Series of Fractions, decreasing in a double Propo tion $\frac{1}{2}$ $\frac{1}{4}$, $\frac{1}{8}$ $\frac{1}{16}$ $\frac{1}{32}$, *&c. ad infinitum*, if you take them back wards, you may justly reckon a Cypher or 0, for the fir Term (for between $\frac{1}{2}$ and 0 there may be an infinite Numbe of such Terms) and the infinite Sum of these Terms will b precisely equal to Unity; for subtracting the first 0, from th last $\frac{1}{2}$, and the remainder $\frac{1}{2}$ being divided by the name of th Reason lessened by 1, *i. e.* by I. which divides nothing, th Quotien

Quotient $\frac{1}{2}$ is the Sum of all the Terms excepting the last, by Prop. 21. and so the last $\frac{1}{2}$ being added, the Sum of all in that Series will be 1. Now if the last is not $\frac{1}{2}$ but 1, the Sum of all will necessarily be 2; if 2 be the last, the Sum of all will be 4; in a word, it will be always double the last Term.

III. And since in this case the Sum of all the precedent Terms is equal to the last Term, the one being subtracted from the other, there will remain nothing, *i. e.* $\frac{1}{2}-\frac{1}{4}-\frac{1}{8}-\frac{1}{16}-\frac{1}{32}$, &c. in *Infinitum*, is $=0$, and also $1-\frac{1}{2}-\frac{1}{4}$, &c. or $2-1-\frac{1}{2}-\frac{1}{4}$. &c. $=0$.

IV. In like manner the Sum of infinite Fractions decreasing in triple Reason in an infinite Series ($\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}$, &c.) will be equal to $\frac{1}{2}$: for if from the last $\frac{1}{3}$ (again in an inverted Order) you subtract the first 0, and the Remainder $\frac{1}{3}$ be divided by the name of the Reason lessen'd by Unit, *i. e.* by 2, the Quotient $\frac{1}{6}$ will be the Sum of all the antecedent Terms, and adding to this last $\frac{1}{3}$ or $\frac{2}{6}$ the Sum of all will be $\frac{3}{6}$ or $\frac{1}{2}$.

V. Thus an infinite Series of Fractions decreasing from $\frac{1}{4}$ in a Quadruple Proportion ($\frac{1}{4}+\frac{1}{16}+\frac{1}{48}$ &c.) is equal to $\frac{1}{3}$; for subtracting the first 0 from the last $\frac{1}{4}$, and the remainder $\frac{1}{4}$ being divided by the name of the Proportion., i. e. by 3, you will have $\frac{1}{12}$ the sum of all except the last, and adding also the last $\frac{1}{4}$ or $\frac{3}{12}$, you'l have the whole Sum $\frac{4}{12}$ or $\frac{1}{3}$.

VI. Thus also an infinite Series decreasing from $\frac{1}{5}$ in a Quintuple Proportion ($\frac{1}{5}+\frac{1}{25}+\frac{1}{125}$, &c.) is equal to $\frac{1}{4}$: $\frac{1}{6}+\frac{1}{36}+\frac{1}{216}$, &c. is equal to $\frac{1}{5}$ &c. and so any Series of this kind is equal to a Fraction, whose Denominator is less by an Unit than the Denominator of the last Fraction in that Series.

VII. Generally also, any infinite Series of Fractions decreasing according to the Proportion of the Denominator of the last Term, and having a common Denominator less by an unit than the Denominator of the last Term (*e.* g. $\frac{2}{3}+\frac{2}{9}+\frac{2}{27}$ &c. or $\frac{3}{4}+\frac{3}{16}+\frac{3}{64}$, &c. or $\frac{4}{5}+\frac{4}{25}+\frac{4}{125}$, &c.) is equal to Unity, after the same way as the Series Consect. 2. which may be comprehended under this kind, and which may be demonstrated in all its particular cases by the same method we have hitherto made use of, or also barely substituted from Consect. 4, 5, and 6. For since $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}$, &c. is equal to $\frac{1}{2}$; $\frac{2}{3}+\frac{2}{9}+\frac{2}{27}$ will be equal to $\frac{2}{2}$, or 1, and so in the rest.

VIII. Particularly the sum of $\frac{3}{27}+\frac{3}{288}+\frac{3}{1152}$, &c, decreasing in a Quadruple Proportion, is equal to $\frac{1}{18}$; and the sum of $\frac{3}{64}+\frac{3}{256}+\frac{3}{1024}$, &c. is equal to $\frac{1}{16}$; and the sum of $\frac{7}{64}+\frac{7}{512}+\frac{7}{4096}$, &c. decreasing in Octuple Proportion, is equal to $\frac{1}{8}$: For subtracting the first Term o, and dividing the remainder by the name of the Reason lessen'd by 1, *i.e.* by 3, the Quotient $\frac{1}{72}$ gives the sum of all except the last. This therefore (*viz.* $\frac{3}{72}$) being added, the sum of all will be $\frac{4}{72}$ or $\frac{1}{18}$: In like manner $\frac{7}{64}$ being divided by the name of the Reason lessen'd by Unity, the Quotient will give $\frac{1}{64}$, and adding the last, the sum of all will be $\frac{8}{64}$, *i. e.* $\frac{1}{8}$. So that hence it is evident, that $\frac{1}{18}-\frac{3}{72}-\frac{3}{288}-\frac{3}{1152}$, &c. or $-\frac{1}{18}+\frac{3}{72}+\frac{3}{288}+\frac{5}{1152}$, &c. *in Infinitum*, will be equal to nothing; also $\frac{1}{8}-\frac{7}{64}-\frac{7}{512}-\frac{7}{4098}$ &c. $=$ o.

IX. *The Sum of a simple Arithmetical Progression* (i. e. *ascending by the Cardinal Numbers*) *continued from* 1, ad Infinitum, *is Subduple of the Sum of the same number of Terms, each of which is equal to the greatest; or on the contrary, this latter Sum is double of the former.* We might have subsumed this in *Consect.* 4. *Prop.* 16. for, prefixing a Cypher before Unity, it will be a case of that *Consectary*, the Sum of the Progression remaining still the same. But that this is true, in an infinite Series beginning from Unity (for in a finite or determinate one, the proportion of the Sum is always less than double, tho it always approaches to it, and comes so much the nearer by how much greater the Series is) we shall now thus Demonstrate: To the Sum of three Terms, 1, 2, 3, *i. e.* 6, the sum of as many equal in number to the greatest, *i. e.* 9, has the same Proportion as 3 to 2; but to the sum of six Terms, 1, 2, 3, 4, 5, 6, *i. e.* 21, the sum of as many equal to the greatest, *i.e.* 36, has the same proportion as 3 to $1+\frac{3}{4}$, that is, as 3 to $2-\frac{1}{4}$, the decrease being $\frac{1}{4}$: but to the sum of 12 Terms, which may be found by *Consect.* 1. *Prop.* 16. $=78$, the sum of so many equal to the greatest, *viz.* 144. has the same proportion (dividing both sides by 48) as 3 to $1\frac{30}{48}$, *i.e.* 3 to $1+\frac{1}{2}+\frac{1}{8}$ (for 24 make $\frac{1}{2}$, and the remainder $\frac{6}{48}$ is the same as $\frac{1}{8}$) that is, as 3 to $2-\frac{1}{4}-\frac{1}{8}$, the decrement being now $\frac{1}{8}$. Since therefore, by doubling the number of Terms onward, you'l find the decrement to be $\frac{1}{16}$, and so onwards in double Proportion; the sum of an infinite Number of such Terms, in Arithmetical Progression, equal to the greatest

will be to the ſum of the Progreſſion from 1, *ad Infinitum*, as 3 to $2 - \frac{1}{4} - \frac{1}{8} - \frac{1}{16}$, &c. that is, by *Conſect.* 2 and 3, as 3 to $2 - \frac{1}{2}$, that is, as 3 to $1\frac{1}{2}$, or as 2 to 1. Q. E. D.

X. The Sum of any Duplicate Arithmetical Progreſſion (*i. e.* a Progreſſion of Squares of whole numbers aſcending) continued from 1 *ad Infinitum*, is ſubtriple of the Sum of as many Terms equal to the greateſt as is the number of Terms: For any ſuch finite Progreſſion is greater than the ſubtriple Proportion, but approaches nearer and nearer to it continually, by how much the farther the Series of the Progreſſion is carried on. Thus the Sum of 3 Terms 1, 4, 9 = 14 is to thrice 9 = 27 as $1\frac{5}{9}$, or $1\frac{10}{18}$, or $1 + \frac{1}{2} + \frac{1}{18}$ to 3 (dividing both ſides by 9,) the Sum of ſix Terms, 1, 4, 9, 16, 25, 36, *viz.* 91. to ſix times 36, *i. e.* to 216 (dividing both ſides by 72) is as $1 + \frac{1}{4} + \frac{1}{72}$ to 3; and the Sum of 12 Terms 650, to 12 times 144. *i. e.* to 1728 (dividing both ſides by 576) is as $1 + \frac{1}{8} + \frac{1}{288}$ to 3, &c. the Fractions adhering to them thus conſtantly decreaſing, ſome by their half parts, others by three quarters (for $\frac{1}{18}$ is $\frac{4}{72}$; therefore the firſt decrement is $\frac{3}{72}$ and $\frac{1}{72}$, is $\frac{4}{288}$; therefore the ſecond decrement is $\frac{3}{288}$, &c.) Wherefore the Sum of the Infinite Progreſſion will be to the Sum of the like number of Terms equal to the greateſt, as

$$\begin{array}{l} 1 + \frac{1}{2} + \frac{1}{8} \\ \quad - \frac{1}{4} - \frac{3}{72} \\ \quad - \frac{1}{8} - \frac{3}{288} \\ \quad - \frac{1}{18}\text{ \&c. } \frac{3}{1152}\text{, \&c.} \end{array}$$

to 3, that is, by Conſect. 3 and 8, as 1 to 3. Q. E. D.

XI. The Sum of a triplicate Arithmetical Progreſſion (*i. e.* aſcending by the Cubes of the Cardinal Numbers) proceeding from 1 thro' 27, 64, &c. *ad Infinitum*, is Subquadruple of the Sum of the like number of Terms equal to the greateſt. For the Sum of 4 Terms, 1, 8, 27, 64, *i.* 100, to 4 times 64, *i. e.* 256 (dividing both ſides by 64) will be found to be as $1 + \frac{1}{2} + \frac{1}{16}$ to 4; but the Sum of 8 Terms, 1, 8, 27, 64, 125, 216, 343, 582, *i. e.* 1296 to 8 times 512, that is, 4096 (dividing both Sides by 1024) will be found to be as $1 + \frac{1}{4} + \frac{1}{4}$ to 4, &c. The adhering Fractions thus

constantly decreasing, the one by their $\frac{1}{2}$ part, the others by $\frac{1}{4}$ (for $\frac{1}{16}$ is $\frac{4}{64}$, and $\frac{1}{64}$ is $\frac{4}{256}$, &c. Wherefore the Sum of the Infinite Progression will be to the Sum of a like (Infinite) number of Terms, equal to the greatest, as

$$1 + \frac{1}{2} \qquad + \frac{1}{8}$$
$$- \frac{1}{4} \qquad - \frac{3}{64}$$
$$- \frac{1}{8}, \&c. - \frac{3}{256}, \&c. \text{ to } 4;$$

that is, by Consect. 3 and 8, as 1 to 4. Q. E. D.

XII. The Sum of an Infinite Progression, whose greatest Term is a Square Number, the others decreasing according to the odd numbers 1, 3, 5, 7, *&c.* is in Subsesquialteran Proportion of the Sum of the like number of equal Terms, *i. e.* as 2 to 3. For the Sum of three such Terms, *e. g.* 9, 8, 5, *i. e.* 22 to thrice 9, *i. e.* 27. is (dividing both sides by 9) as $2\frac{4}{9}$, *viz.* $\frac{8}{18}$ to 3, or $2 + \frac{1}{2} - \frac{1}{18}$ to 3. But the Sum of six such Terms, 36, 35, 32, 27, 20, 11, *i. e.* 161, to six times 36, *i. e.* 216 (dividing both sides by 72) is as $2 + \frac{1}{4} - \frac{1}{72}$, &c. the adhering Fractions thus always decreasing, some by $\frac{1}{2}$, others by $\frac{3}{4}$, as above in Consect. 10. Wherefore the Sum of the Infinite Progression will be to the Sum of the like number of Terms equal to the greatest, as

$$2 + \frac{1}{2} - \frac{1}{18}$$
$$- \frac{1}{4} + \frac{3}{72}$$
$$- \frac{1}{8} + \frac{3}{288}, \&c. \text{ to } 3,$$

i. e. by Consect. 3 and 8, as 2 to 3. Q. E. D.

SCHOLIUM II.

THus we have, after our method, demonstrated the chief Foundations of the *Science* or *Method*, or *Arithmetick* of *Infinites*, first found out by Dr. *John Wallis*, Savilian Professor of Geometry at *Oxford*, and afterwards carried further by *Detlerus Cluverus*, and *Ismael Bullialdus*. And from these Foundations we will in the following Treatise demonstrate, and that directly and *à priori*, in a few Lines, the chief Propositions of Geometry

Geometry, which the Antients have ſpent ſo much labour, and composſed ſuch large Volumes to demonſtrate, and that but indirectly neither.

Propoſition XXII.

The Powers of Proportionals whether continuedly or diſcretely, ſuch as the Squares, Cubes, &c. *are alſo Proportional.*

Demonſtration.

Continual Proportionals.					Diſcrete Proportionals.			
	a	ea	e^2a	e^3a	a	ea	b	eb
Squares	aa	e^2a^2	e^4a^2	e^6a^2	a^2	e^2a^2	b^2	e^2b^2
Cubes	a^3	e^3a^3	e^6a^3	e^9a^3	a^3	e^3a^3	b^3	e^3b^3

Q. E. D.

SCHOLIUM.

You founded in this Truth, 1. the Reaſon of the Multiplication and Diviſion of Surd Quantities: For ſince from the Nature and Definition of Multiplication, it is certain, that 1 is to the Multiplier as the Multiplicand to the Product (for the multiplicand being added as many times to it ſelf as there are Units in the Multiplier, makes the Product) if the $\surd 5$ is to be multiplied by $\surd 3$, then as 1 to the $\surd 3$, ſo the $\surd 5$ to the Product; and, by the preſent Propoſition, as 1 to 3, ſo 5 to the □ Product, *i. e.* to 15. Wherefore the Product is $\surd 15$; and ſo the Rule for Multiplying Surd Quantities is this: *Multiply the Quantity under the Radical Signs, and prefix a Radical Sign to the Product.* Likewiſe ſince it is certain from the Nature of Diviſion, that the Diviſor is to the Dividend as 1 to the Quotient (for the Quotient expreſſes by its Units how many times the Diviſor is contained in the Dividend) if the $\surd 15$ is to be divided by $\surd 5$, you'l have $\surd 5$ to the $\surd 15$ as 1 to the Quotient, and, by the preſent Scholium, 5 to 15, as 1 to the □ of the Quotient, *i. e.* to 3. Therefore the Quotient is the Root of 3, and ſo the Rule of dividing Surd Quantities this; *viz.*

(*a*) *Eucl. lib.* 6. *prop.* 22.

Divide the Quantities themselves under the Radical Signs, and prefix the Radical Sign to the Quotient.

II. Hence also flows the usual Reduction in the Arithmetick of Surds, of Surd Quantities to others partly Rational, and on the contrary, of those to the form of Surds, *e. g.* If you would reduce this mixt Quantity $2a\sqrt{b}$, *i. e.* $2a$ multiplied by the $\sqrt{b}$, to the form of a Surd Quantity; which shall all be contained under a Radical Sign; The Square of a Rational Quantity without a Sign $4aa$, if it be put under a Radical Sign, in this form $\sqrt{4aa}$, it equivalent to the Rational Quantity $2a$; but the $\sqrt{4aa}$ being multiplied by $\sqrt{b}$ makes $\sqrt{4aab}$. by N° 1. of this Scholium. Therefore $\sqrt{4aab}$ is also equivalent to the Quantity first proposed $2a\sqrt{b}$. Reciprocally therefore, if this form of a meer Surd Quantity $\sqrt{4aab}$, is to be reduced to one more Simple, which may contain without the Radical Sign whatever is therein Rational, by dividing the Quantity comprehended under the sign $\sqrt{}$ by some Square or Cube, &c. as here by $4aa$, (*i. e.* $\sqrt{4aab}$ by $\sqrt{4aa}$, i. e. $2a$) the Quotient will be $\sqrt{b}$, which multiplied by the Divisor $2a$, will rightly express the proposed Quantity under this more simple Form $2a\sqrt{b}$. Which may also serve further to illustrate the Scholia of Prop. 7. and 10.

Proposition XXIII.

If there are four Quantities Proportional, (a, ea, b, eb) they will be also Proportional,

(α) Eucl. 15, 16. v. 9 10, 13, vi. (β) 18, v. (γ) 17, v.

1. Inversly. ea to a as eb to b.
2. Alternatively, (α) a to b as ea to eb.
3. Compoundedly, (β) $a+ea$ to ea, so $b+eb$ to eb.
4. Conversly, $a+ea$ to a as $b+eb$ to b.
5. Dividedly, (γ) $a-ea$ to $\left\{\begin{array}{l} ea \\ \text{or } a \end{array}\right.$ as $b-eb$ to $\left\{\begin{array}{l} eb \\ \text{or } b \end{array}\right.$
6. (α) By a Syllepsis, a to ea as $a+b$ to $ea+eb$.
7. By a Dialepsis, a to ea as $a-b$ to $ea-eb$.

Which are all manifest, by comparing the Rectangles of the Means and Extremes according to to *Prop.* 19. and its *Consect.* 1.

or by dividing any of the Consequents by their Antecedents, according to *Def.* 31.

Proposition XXIV.

IF in a (β) double Rank of Quantities you have
as a to ea, and also as ea to oa,
so b to eb, so eb to ob,
then you'l have also by proportion of Equality orderly placed,
as the first a, to the last oa, in the first Series;
so the first b, to the last ob, in the second Series.
Which is manifest from the Terms themselves.

Proposition XXV.

BUt (γ) if they are disorderly plac'd
as oa to ea * as ea to a †
† so eob to ob so ob to eb, * you'l have here
again by proportion of Equality,
as the first oa to the last a, in the first Series;
so the first eob to the last eb, in the second Series.
As is evident from the Rectangles of the Extremes and Means, as also from the very Terms.

(α) *Eucl.* 1, 12. v. 5. 6. 12. vii.
(β) *Eucl.* 3, 20, 22. *lib.* v. 14. vii.
(γ) *Eucl.* 21, 23. *lib.* v.

Proposition XXVI.

IF (α) as the whole ea to the whole a, so the part eb to the part b; then also will
the Remainder $ea—eb$ to the Remainder $a—b$, as the Whole ea to the Whole a.
This is evident from the Rectangle of the Extremes and Means, both which are $eaa—eab$. Q.E.D.

Proposition XXVII.

REctangles or Products having one common Efficient or Side, are one to another as the other Efficients or Sides.

Demonstration.

Suppose the Products to be ab and ac, having the common Efficient a; I say they are

as b to c, so ab to ac.

Which is evident at first sight, by comparing the Products of the Extremes and Means, and also fully shews, that other way of proving Proportionality, whereby by dividing the Consequents by their Antecedents, the identity or sameness of the Quotients are wont to be demonstrated.

SCHOLIUM I.

I. THe Reduction of Fractions either to more compounded or more simple ones is founded on this Theorem; on the one hand by multiplying, on the other by dividing, by the same quantity, both the Numerator and the Denominator, as, *e. g.* $\frac{c}{b}$ and $\frac{ac}{ab}$ and $\frac{eac}{eab}$, $\frac{1}{3}$, $\frac{3}{9}$, $\frac{4}{12}$, &c. are in reality the same Fractions. And

II. The Reduction of Fractions to the same Denomination, as if $\frac{b}{c}$ and $\frac{a}{d}$ are to be changed into two others that shall have same Denominator; this is to be done by multiplying the Denominators together for a new Denominator, and each Numerator by the Denominator of the other for a new Numerator, and you'l have for the two Fractions above ——— $\frac{bd}{cd}$ and $\frac{ac}{cd}$.

(α) *Eucl.* 5 & 19. *lib.* v. 7 & 11. *l.* 7. (β) *Besides several other Prop. see also the* 17 & 18 *lib.* vii.

SCHOLIUM II.

WE will here for a conclusion of Proportionals, shew the way of cutting or dividing any Quantity in *Mean and Extreme Reason*, *viz.* if for the greater Part you put x, the less will

will be $a-x$; and so by Hypoth. these three, a, x and $a-x$, will be proportional, by *Def.* 34. Therefore by *Prop.* 17. the Product of the Extremes $aa-ax=$ to the Square of the Mean xx, and (adding on both sides ax) $aa=xx+ax$; and moreover adding on both sides $\frac{1}{4}aa$, you'l have $\frac{5}{4}aa=xx+ax+\frac{1}{4}aa$. Now this last Quantity, since it is an exact Square, whose Root is $x+\frac{1}{2}a$, you'l have $\sqrt{\frac{5}{4}aa}=x+\frac{1}{2}a$, and (subtracting from both sides $\frac{1}{1}a$) $\sqrt{\frac{5}{4}aa}-\frac{1}{2}a=x$.

Now therefore we have a Rule to determine the greater part of a given Quantity to be divided in Mean and Extreme Reason, *viz.* if the given Quantity be a Line, *e. g.* $AB=a$ (*Fig.* 58.) join to it (α) at Right Angles $AC=\frac{1}{2}a$: Wherefore by the Theorem of *Pythagoras* from *Schol. Definit.* 13. the Hypothenuse CB, or, which is equal to it, $CD=\sqrt{\frac{5}{4}aa}$; and consequently $AC=\frac{1}{2}a$ being taken out of CD, the Remainder AD, or AE, which is equal to it, will be $=x$, the greatest part sought; according to *Euclid*, whose Invention this first Specimen of Analysis, by way of Anticipation, reduces to its original Fountain. As for Numbers (tho none accurately admits of this Section) the sense of the Rule, or which is all one as to the thing it self, is this: Add the Squares of a whole Number and its half, and subtract the said half from the Root of the Sum (which can't be had exactly, since it is $\sqrt{\frac{5}{4}}$.

(α) *Eucl.* 11. *lib.* 11. & 30. *lib.* vi.

CHAP. V.

CAAP. V.

Of the Proportion or Reasons of Magnitudes of the same kind in particular.

Proposition XXVIII.

TRiangles and Parallelograms, also Pyramids and Prisms and Parallelepipeds, lastly Cones and Cylinders, each kind compared among themselves, if they have the same Altitude, are in the same Proportion to one another as their Bases.

Demonstration.

This and the following Proposition might have been by a bare Subsumption added, as *Consectarys* to the precedent; for the Altitudes in the one, and Bases in the other, may be looked on as common Efficients, and the Magnitudes mentioned as their Products: But for the greater distinction sake, we will thus Demonstrate them more particularly.

(α) *Eucl. Prop.* 1. *lib.* vi. (β) *Prop.* 5. 6. *lib.* xii. 25, 32. xi. & *Cons.* 30 & 31 *of the same* (γ) *Prop.* 11. *lib.* xii. (δ) *Prop.* 35, 36, 37, 38, 39, 40, *lib.* *lib.* 1 & 29, 30, 31. *lib.* xi.

I. If the equal Altitudes of two Triangles, or (α) two Parallelograms, are called *b* and the Base of the one *a*, and of the other *ea*; these Products will be *ba* and *bea*, the other $\frac{1}{2}$ *ba* and $\frac{1}{2}$ *bea*, by *Def.* 28. *Schol.* 2.

II. Likewise the equal Altitudes of two Prisms (β) or Pyramids, may be called *b*, and the Proportion of their Bases expressed by *a* and *ea*; and the Prisms will be among themselves as *ba* to *bea*, and the Pyramids as $\frac{1}{3}$*ba* to $\frac{1}{3}$*bea*, by the said *Schol. Num.* 3.

III. There is also the same Proportion of Cylinders and Cones as of Pyramids and Prisms, by *Consect.* 4. *Definit.* 17. But,

as a to ea so is ba to bea.
—$\frac{1}{2}ba$ to $\frac{1}{2}bea$.
—$\frac{1}{3}ba$ to $\frac{1}{3}bea$. Q. E. D.

CONSECTARY.

THerefore Magnitudes of the same kind upon the same or equal Bases (♪) and of the same heighth, are equal among themselves, and the contrary.

Proposition XXIX.

TRiangles and Parallelograms, Pyramids and Prisms and Parallelepipeds, Cones and Cylinders, being on equal Bases, are in the same Proportion as their heighths. (*)

Demonstration.

Let all their Bases be called a, and the Proportions of their Heighths be as b to eb: Therefore, 1. the Parallelograms, Parallelepipeds and Cylinders, are one to the other of the same kind, as ba to eba; the Triangles as $\frac{1}{2}ba$ to $\frac{1}{2}eba$; the Pyramids and Cones as $\frac{1}{3}ba$ to $3\,eba$, by *Def.* 28. *Schol.* 2. But,
as b to eb, so is ba to eba.
and $\frac{1}{2}ba$ to $\frac{1}{2}eba$.
and $\frac{1}{3}ba$ to $\frac{1}{3}eba$. Q. E. D.

Proposition XXX.

EQual (‖) *Triangles, Parallelograms, Prisms, Parallelepipeds, also equal Pyramids, Cones, and Cylinders, have their Bases and Heighths reciprocally Proportional.*

Demonstration.

For if for the equal Triangles you put $\frac{1}{2}ab$, for the Cones and Pyramids $\frac{1}{3}ab$, and for the rest ab;

whether the Bases of the equal Quantities are supposed to be a, and so the Altitudes on

(*) *Schol prop.* 1.*l.*6,12,13,14
‖*lib.*6.*prop.*14. 15.*l.*11.34. *l.* 12.11, *and its Coroll. also Prop.* 15.

on both sides b; or if the Base of the one be a and b the Altitude, but the Base of the other b and the Altitude a, you'l certainly have eitherways,

as a to a, so Reciprocally b to b;

the Base of the former to the Base of the latter, as the Altitude of the latter to the Altitude of the former, or,

as a to b, so Reciprocally a to b. Q.E.D.

CONSECTARY.

AND those Magnitudes of the same kind, whose Bases and Altitudes are thus Reciprocal, are equal by *Prop.* 18. for the Product or Rectangle of the Extremes is ab, and that of the Means ba.

Proposition XXXI.

Triangles, Parallelograms, Prisms, Parallelepipeds, Pyramids, Cones and Cylinders, each kind compared among themselves, are in the Proportion compounded of the Proportion of their Altitudes and Bases. (a)

Demonstration.

Suppose the Base of the one to be a, and the other ea, and the Altitude of the one b, of the other ib; therefore the one will be to the other,

as ab to $eiab$,

or $\frac{1}{2}ab$ to $\frac{1}{2}eiab$,

or $\frac{1}{3}ab$ to $\frac{1}{3}eiab$; *i. e.* every where as a to eia, *i. e.* in Proportion compounded of a to ea, and of b to ib, by *Consect.* 2. *Def.* 34. Q.E.D.

SCHOLIUM.

FRom what we have hitherto Demonstrated, we may not only make an estimate of Magnitudes of the same kind compared together, which is easie to any one who attentively considers them; but also with *F. Morgues*, deduce a General Rule of expressing the Proportions of any Rectilinear Planes or Solids, contained under Planes

(a) *Prop.* 23. *lib.* 6.

Plane Surfaces, by the proportion of one Right Line to another. For ſince the one may be reſolved into Triangles, and the other into Pyramids, having firſt two Rectilinear Planes given and thus reſolved, upon a Right Line I make the △ *abc* (*Fig.* 49.) Equal to one of the Triangles of either of the Planes *e. g.* to ABC; then having drawn the Parallel *cm*, if the △BCD has the ſame Altitude with the former, you need only joyn the Baſe BC to the Baſe *a b*. But if the Altitude DS is greater than the Altitude of the other *e. g.* by $\frac{1}{5}$, then you muſt make *bf* equal to the Baſe *bc* augmented by a fifth part, and the Triangle *bcf* will = BCD, and the whole *acf* = to the Rectilinear Figure ABCD. If now therefore I likewiſe make another Triangle *ghi* equal to another Rectilinear Figure between the ſame Parallels, then will the △ *acf* be to the △ *ghi*, that is, the Right Lined Figure ABCD to the Right Lined Figure FGHIK, as *af* to *gh*, by *Prop.* 28. 2. Having 2 Right Lined Solids given, and having reſolved them into Triangular Pyramids, they may be transferr'd between 2 parallel Planes, *viz.* by augmenting or diminſhing their Triangular Baſes reciprocally, according to the exceſs or defect of their Altitudes, as was done above with the Linear Baſes; then thoſe Triangular Baſes on both ſides may be converted into one Triangular Baſe, and conſequently each Solid into a Pyramid equal to it ſelf; which two Pyramids will be one to the other as their Triangular Baſes. And becauſe the Proportions of theſe Baſes may be reduced to the Proportion of two Lines each to the other, by N° 1. of this; therefore alſo the Reaſon or Proportion of the two Solids may be expreſſed by the Proportion of two Lines. Q.E.D.

Propoſition XXXII.

CIrcles (β) *are in the ſame Proportion to one another as the Squares of their Diameters.*

Demonſtration.

Suppoſe *a* to be the Diameter of one Circle, and *b* of another; then by *Definit.* 31. *Conſect.* 1. the Area of the one will be $\frac{1}{4}$ *eaa*, and that of the other $\frac{1}{4}$ *ebb*. But as *aa* to *bb* ſo is $\frac{1}{4}$ *eaa* to $\frac{1}{4}$ *ebb*, by *Conſect.* 1. *Prop.* 19. Q. E. D.

(β) *Eucl. Prop.* 2. *l.* 12.

CON-

CONSECTARY I.

THe ſame will in like manner be manifeſt of like Sectors
Circles, while for the parts of the Periphery you put *i*
and *i b*, as for the wholes we put *ea* and *eb*: for thus the Ar
of the one will be $\frac{1}{4}$ *iaa*, and of the other $\frac{1}{4}$ *ibb*.

CONSECTARY II.

CYlinders whoſe Altitudes are equal to the Diameters of th
Baſes, are in proportion to one another as the Cube
their Diameters; for the Cylinders will be $\frac{1}{4}ea^3$ and $\frac{1}{4}eb^3$,
Cubes a^3 and b^3.

CONSECTARY III.

HEnce alſo (whatever the Reaſon of the Sphere is to the
Cylinder of the ſame Diameter and Heighth; which
will hereafter Demonſtrate, and which in the mean while
will denote by the name of the Reaſon *y*) I ſay, hence Sphe
which have the ſame Proportion to one another as theſe Cyl
ders (*viz.* as $\frac{1}{4}ea^3$ to $\frac{1}{4}eb^3$, ſo $\frac{1}{4}yea^3$ to $\frac{1}{4}yeb^3$) will alſo (by C
ſect. 1.) be in the ſame proportion as the Cubes, a^3 to b^3;
is alſo evident from theſe Terms themſelves.

Propoſition XXXIII.

T*HE Angle* (β) *at the Center of any Circle* ACB (*Fig.* 60)
to an Angle at the Circumference which has the ſame Arch
its Baſe ADB, *as* 2 *to* 1.

Demonſtration.

(α) *Prop.* 18 *lib.* 12.
(β) *Eucl. Prop.* 20. *l.* 3.

The truth of this has already appear'd fro
Schol. Definit. 10. N° 3. but here we will demo
ſtrate it otherwiſe in its three Caſes, after *Eu*
clids way. In the firſt Caſe DE being conceive
Parallel to CB, by *Def.* 11. *Conſect.* 1 and 2. th
External Angle ACB is = to the Internal An
gl

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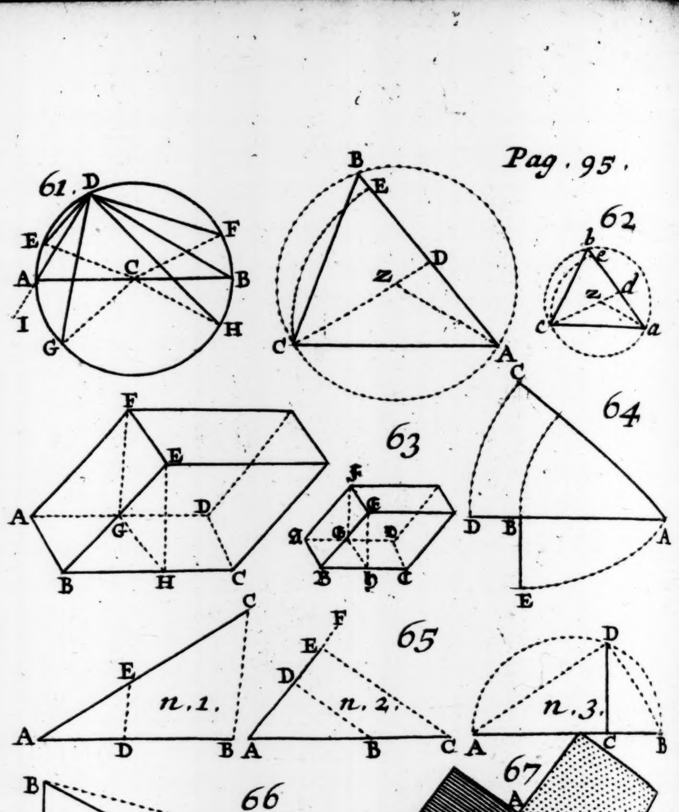


gle ADE, and the Angle BDE, is equal to the alternate Angle BCD, *i. e.* to the other at the Base CDB, by *Consect.* 2. *Definit.* 13. Therefore BDE is as 1, and CDE, *i. e.* A C B as 2.

In the second Case the whole ECB is double of the whole EDB, and the subtracted Angle ECA is double of the subtracted Angle EDA, by Case 1. Therefore the Remainder ACB is also double of the Remainder ADB, by Prop. 26. In the third Case the part ECA is double of the part EDA, and also the part ECB is double of the part EDB, by Case 1. Therefore the whole ACB is double the whole ADB. Q.E.D.

CONSECTARYS.

I. HEnce all Angles ADB (α) in the same Segment are equal, and the Angle ADB (*Fig.* 61.) in a Semicircle is a Right one; because the Aperture at the Center answering to it, ACB contains two Right Angles: The Angle in a less Segment than a Semicircle EDF, is greater than a Right one; because the Aperture at the Center EGHFC answering to it, comprehends more than two Right Angles. An Angle, lastly, in a Segment greater than a Semicircle GDH, is less than a Right one; because its double at the Center GCH is less than two Right ones. All which we have already otherwise demonstrated in *Schol. Def.* 10. N° 6.

II. All the three Angles (β) of any Triangle ABD taken together, are equal to two Right ones; because they are the half of the three at the Center C, which always make 4 Right ones, by *Definit.* 8. *Consect.* 2.

(α) *Eucl. prop.* 21. 27. 31. *lib.* 3.
(β) *prop.* 32. *l.* 1

III. Therefore any external Angle IAB, is equal to the two Internal opposite ones at B and D; because that, as well as they with the other contiguous to them BAD, make two Right ones, by *Consect.* 1. of the same *Definit.*

IV. And the greatest Side of a Triangle, because it insists on a (α) greater Arch of a Circumscribed Circle, does also necessarily subtend a greater Angle, by vertue of *Consect.* 1. hereof.

Proposition XXXIV.

IN *Equiangular Triangles* (*ACB and* abc, *Fig.* 62.) *the Sides a-bout the equal Angles are Proportional,* viz. *as AB to BC, so ab to* bc, *and as BC to CA so is* bc *to* ca. *&c.* (β)

Demonstration.

For having described Circles thro' the Vertex of each Triangle, according to *Consect.* 6. *Definit.* 8. by reason of the supposed equality of the Angles A and *a*, B and *b*, C and *c*, the Arches also AB and *ab*, &c. will necessarily agree in the number of Degrees and Minutes, by the foregoing 33 *Prop.* and so also the Chords AB and *ab*, BC and *bc*, &c. will agree in the number of Parts of the Radius or whole Sine ZA and *za*, by *Consect.* 2. *Definit.* 10. Wherefore as many such Parts as AB has, whereof *az* has also 10000000, so many such also will *ab* have, whereof *az* has also 10000000, *&c.* Therefore AC to CB as *ac* to *cb*, &c. Q. E. D.

CONSECTARYS.

I. WHerefore by the same necessity the Bases of such Triangles AB and *ab*, will be proportional to their Altitudes CD and *cd*, as being Right Sines of the like Arches CB and *cb*, or rather CE and *ce*; and so for similar or like Triangles (and consequently also Parallelograms) we may rightly suppose that their Bases are as *a* to *ea*, and their Heighths as *b* to *eb*; tho we must not immediately conclude on the contrary, that because their Bases and Altitudes are so, therefore they are Similar.

II. As also in Similar Parallelepipeds it will be manifest to any attentive Person, that the Bases are in a duplicate Proportion of the Altitudes. For since the Planes of Similar Solids are equal in number, and Similar each to the other, if for 𝔄𝔅 (*Fig.*63.) we put *a*, and for 𝔅ℭ *b*, AB will = *ea* and BC = *eb*; and so that Basis will be to this as *ab* to *eeab*. Moreover having let fall the Perpendiculars EH and 𝔈𝔥, the Triangles

FBH

BH and EBH are similar, and by putting *c* for BE, BE will be *e c*, putting also *d* for EH, EH will consequently be *e d*. But the Reason of the Base *a b* to the Base *e e a b*, is duplicate of the Reason of *d* to *e d*, by *Def.* 34. Wherefore in Similar Parallelepipeds we may rightly suppose, that their Bases are as *a b* to *eeab*, or as *a* to *eea*, and their Altitudes as *d* to *ed*.

SCHOLIUM I.

FRrom this Proposition flows first of all the chiefest part of Trigonometry for the Resolution of Right Angled △△: For since in any Right Angled Triangle, if one side, *e. g.* AB (*Fig.* 64.) be put for the whole Sine, the other BC will be the Tangent of the opposite Angle at A (and in like manner if CB be the whole Sine, BA will be the Tangent of the Angle C;) but if the Hypothenuse AC be made Radius or whole Sine, then the Side BC will be the Right Sine of the Angle A, or the Arch CD described from the Center A, and AB the Right Sine of the Angle C, or the Arch AE, described from the Center C, (we will omit mentioning the Secants, because the business may be done without them) which all follow from *Def.* 10. Wherefore you may find,

I. The Angles.

1. From the Sides	by inferring	As one leg to the other, so the whole Sine to the Tangent of the Angle opposite to the other Leg.
2. From the Hypoth, & one side.		As the Hyp. to the W.S. (whole sine) so the given leg to the S. of the opp. angle

II. The Sides.

1. From the Hypoth. and Angles:	As the W. S. to the Hypoth. so the Sine of the Angle, opposite to the Leg sought, to the Leg it self.
2. From one Leg and the Angles:	As the W. S. to the given Leg, so the Tangent of the Angle adjacent to it, to the Leg sought.
3. From the Hypoth. and one of the Sides:	Having first found the Angles, it's done by the 2, 1. or by the *Pythagorick* Theorem

III. The Hypothenuse.

1. From the Angles and one of the Legs.	As the S. of the Angle, opposite to the given Leg, to that Leg, so the W. S. to the Hypoth.
2. From the Legs given;	Having first found the Angles its done by the 1. or by the *Pythagorick* Theorem.

III. Inversly also, if two Triangles ABC and 𝔄𝔅ℭ (in the Figure of the present Proposition) have one Angle of one equal to one Angle of the other (*e.g.* B and 𝔅) and the Sides that contain these equal Angles proportional (*viz.* as AB to BC so 𝔄𝔅 to 𝔅ℭ) then the other Angles (A and 𝔄, C and ℭ) will be also equal, and the Triangles similar (α) for to the like Chords AB and 𝔄𝔅, BC and 𝔅ℭ, there answer by the Hypoth. like or similar Arches, *i. e.* equal in the number of Degrees and Minutes; and to these also there answer equal Angles both at the Periphery and Center.

IV. (*Fig.* 65. N° 1.) If (β) the Sides of the Angle BAC are cut by a Line DE, parallel to the Base BC, the Segments of those Sides will be proportional, *viz.* AE to EC as AD to BD; for by reason of the Parallelism of the Lines BE and BC, the Triangles ADE and ABC are Equiangular: Therefore as the whole BA to the whole AC, so the part AD to the part AE, and consequently also the remainder EC to the remainder DB as the part EA to the part AD, by *Prop.* 26. and alternatively by *Prop.* 24. EC will be to EA as BD to AD.

SCHOLIUM II.

THere are several useful Geometrical Practices depending on this *Consectary* and its *Proposition.* 1. That (γ) whereby we are taught to cut off any part required, *e. g.* $\frac{1}{3}$ from a given Line AB, and so generally to cut or divide any given Line AC, in the same proportion as any other given Line, is supposed to be divided in D, (and consequently into as many equal parts as you please;) *viz.* if in the first Case, having drawn any Line AF, you take AD 1, and make DB 2, and having joined CB, draw the

(α) *Eucl prop.* 6. *lib.* 6.
(β) *Eucl.* 2. *lib.* 6.
(γ) *Eucl. l.* 9 & 10. *l.* 6.

the Parallel LE: for as AD to DB so is AE to AC; that is, as 1 to 2, by this 4th *Consect.* therefore AE is one third of the whole AC, *&c.*

II. A Rule (α) to find a third Proportional to the 2 Right Lines given AB and BC (N° 2. *Fig.* 65.) (or a fourth to three given;) if, *viz.* having drawn AF at pleasure, you make AD equal to BC, and Joining DB, draw the Parallel EC: For as AB to BC, so AD *i. e.* BC) to DE. Now if AD be not equal to BC but to another (*viz. a*) third Proportional, then by the same Reason DE will be a fourth Proportional.

III. Another Rule (β) to find a mean Proportional between two Right Lines given AC and CB; which is done by joining both the Lines together, and from the middle of the whole AB describing a Semicircle, and from C erecting the Perpendicular CD: For since the Angle ADB is a Right one, by *Consect.* 1. of the preceding *Proposition*, and the two Angles at C are Right ones, and those at A and B common to the whole Triangle ADB, and to the two partial ones ACD and BCD, these two will be Equiangular and Similar to the great one, and consequently to one another: Therefore by the present *Proposition*, as AC to CD, so CD to CB, Q. E. D. and also as AB to BD so BD to BC, and as AB to AD so AD to AC, *&c.*

IV. The Analytical Praxis of multiplying and dividing Lines by Lines, so that the Product or Quotient may be a Line; and also the way of Extracting Roots out of Lines: Which *Des Cartes*, gives us, p. 2. of his Geom. *viz.* assuming a certain Line for Unity, *e. g.* AB (in *Fig.* 65. N° 2.) if AC is to be multiplied by AD, having joined BD, and drawn the Parallel CE, the Product will be AE; for it will be as 1 to the Multiplier AD, so the Multiplicand AC to the Product AE; or if AE is to be divided by AC, having joined EC and drawn the Parallel BD, the Quotient will be AD; (for AC the Divisor, will be to AE the Dividend, as an Unit AB to the Quotient AD;) all which are evident from the Nature of Multiplication and Division, and the

(α) *Eucl.* 11 & 12, *l.* 6.
(β) *Eucl.* 13. *lib.* 6 & *and Eucl.* 8. *lib.* 6.

 precedent

Precedent Praxes. As also taking CB (in the same *Fig.* N° 3.) for Unity, if the Square Root is to be extracted out of any other Line AC, this being joined to your Unity in one Line AB, and having described thereon a Semicircle, the Perpendicular CD will be the Root sought, as being a Mean Proportional between the two Extremes CB and AC, according to *Prop.* 17.

V. A Right Line AG which divides (α) any given Angle A into two equal Parts (*Fig.* 66.) being prolonged, divides the Base BC proportionally to the Legs of the Angle AB and AC. For having prolonged CA to E, so that AE shall be = to AB, the Angles ABE and AEB will be equal, by *Consect.* 2. *Def.* 1, and consequently also equal to each of the halves of the external Angle CAB, by *Consect.* 3. of the antecedent Proposition. Therefore the lines AG and EB will be parallel, by *Conf.* 1. *Def.* 11. Therefore as AC to AE, *i.e.* to AB, so GC to GB, by *Consect.* 3. of this Proposition. Q.E.D.

VI. Hence also there follows further, by conversion of the last inference, as AC+AB to AC, so GC+GB (*i.e.* BC) to GC; and inversly GC to BC as AC to AC+AB; and lastly alternatively, GC to AC as BC to AC+AB.

N. B. This last Inference follows also immediately from the preceding *Consectary*. For by reason of the Similitude of the △△ ACG and ECB, as GC to AC so BC to CE, *i. e.* to AC +AB.

SCHOLIUM III.

(α) *Eucl.* 3. *lib.* 6.

FRom these two last Consectarys there arise these or two or three Practical Rules, the first whereof shews, how *having the two Legs AB and AC given, and also the Base BC, to find the Segments GC and GB, made by the Bisection of the Intercrural Angle* (*viz.* by this inference, according to *Consect.* 6: As the Sum of the Sides to one Side (*e. g.*) AC;) so the Sum of the Segments of the Base, *i. e.* the whole Base to one of the Segments, viz. that next the said Side GC. 2. It shews on the contrary, how *having the Base and one of its Segments given, and moreover the Sum of the Sides, to find separately the Side AC next the known Segment*

by inferring as the Sum of the Segments, or the Base BC to the Sum of the Sides, so the given Segment GC to the sought AC: or also, 3dly, *Having only the Base and Sum of the Sides given*, but not the Segment GC, yet to express its Proportion to the next side AC, —— —— *viz*. in the Quantities of the given Terms, by putting (by *Consect*. 6.) for GC the value of the Base BC, and for AC the value of the Sum AB+AC; the great use of which last Rule will appear hereafter in the Cyclometry (or Quadrature of the Circle) of *Archimedes*.

VII. In any Triangle ABC (*Fig*. of the present *Proposition*) the Sides are to one another as the Sines of their opposite Angles: For they are as the Chords of the double Angles at the Center, by *Prop*. 33. therefore they are also one to another as half those Chords, *i. e.* by *Definit*. 10. as the Sines of the half Angles.

SCHOLIUM IV.

HEnce flow two new Rules of Plane Trigonometry, for Oblique-angled Triangles to find, *viz*.

I. The other Angles:

From 2 given Sides, & an Angle opposite to one of them:	by inferring	As the Side opposite to the given Angle to the other Side, so is the Sine of the given Angle to the sine of the angle opposite to the other Side; which being given, the third is easily found.

II. The other Sides:

From one side and the angles given,	As the Sine of the Angle opposite to the given side, to that side; so is the Sine of the Angle opposite to the side sought to the side sought.

So that this way we have reduced all the Cases excepting one of Plane Trigonometry, and consequently all *Euthymetry* to their original Foundations (for in that Case of having two Sides, and the included Angle given, we may find the rest by

the Resolution of the Obliqueangled Triangle into two Right Angled ones; and so it's done by the Rules we have deduc'd in (*Schol.* 1.) I say, excepting one, in which from the three sides of an Obliqueangled Triangle given, you are required to find the Angles: the Rule to resolve which we will hereafter deduce in the 2d *Consect.* of *Prop.* 45. from that Theorem which *Euclid* gives us, *lib.* 2. *Prop.* 13.

VIII. Because in in the Right Angled △ BAC (*Fig.* 67.) BC is to CA as CA to CD, by N° 3. of the 2d *Schol.* of this *Prop.* the □ of CA will be = ▭ CE, by *Prop.* 17. In like manner because as CB to BA so is *B*A to BD; the □ of BA will be = to ▭ BE: Wherefore the two Rectangles BE and CE taken together, that is, the □ of the Hypothenuse BC, will be = to the two □'s BA and CA taken together: Which is the very Theorem of Pythagoras demonstrated two other ways in *Schol.* of *Definit.* 13.

SCHOLIUM V.

This Theorem of *Pythagoras* as it furnishes us with Rules of adding Squares into one Sum, or subtracting one Square from another; so likewise it helps us to some Foundations whereon, among the rest, the structure of the Tables of Sines relies, &c. Whose use we have already partly shewn in *Schol.* 1 and 4. 1. If several Squares are to be collected into one Sum, having joined the Sides of two of them so as to form a Right Angle, *e.g.* A*B* and BC (*Fig.* 68. N° 1.) the Hypothenuse AC being drawn, is the Side of a Square equal to them both; and if this Hypothenuse AC be removed from B to D, and the Side of the third Square from B to E, the new Hypothenuse DE will be the Side of a Square equal to the three former taken together. 2. If the Square of the side MN (N° 2.) is to be subtracted from the Square of the side LM. Having described a Semicircle upon LM, and placed the other MN within that Semicircle, then draw the Line LN and that will be the Side of the remaining Square. 3. Having the Right Sine EG of any Arch ED given (but how to find the Primary Sines we will shew in another place), you may obtain the Sine Complement CG or EF, by the preceding

Numb. viz. by ſubtracting the □ of the given Sine from the □ of the Radius; and moreover the verſed Sine GD by ſubtracting the Sine Complement CG from the Radius CD. 4. The Squares of the verſed Sine GD, and of the Right ſine EG being added together, give the □ of the Chord ED of the ſame Arch, (which all are evident from the *Pythagorick Theorem*) and half of that EH gives the Right Sine of half that Arch. 5. From the Right Sine EG you have the Tangent of that Arch, if you make, as the Sine Complement CG to the Right Sine GE, ſo the whole Sine CD to the Tangent GI. 6. Laſtly, From theſe *Data* you may alſo have the Secants (if required) thus, as the Sine Complement CG to the W. S. CE, ſo the W. S. CD to the Secant CI; or as the Right Sine EG to the W. S. E.C. ſo the Tangent ID to the Secant IC; both which are evident by our 34th Propoſition.

Conſect. 9. If the Quadrant of a Circle (CBEG, Fig. 70.) be inclined to another Quadrant (CADG) and two other Perpendicular Quadrants cut both of them, *viz.* FBAG and FEDG, and the latter do ſo in the extremities of them both) having let fall Perpendiculars from the common Sections E and B, thro' the Planes of the Perpendicular Quadrants, and the inclined Quadrant, (*viz.* on the one ſide EG and BH, as Right Sines of the Segments EC and BC; on the other EI and BK, as Right Sines of the Segments ED and BA) you'l have 2 Triangles EIG and BKH Right Angled at I and K, Equiangular at G and H (by reaſon of the ſame inclination of the Plane CBEGC) and conſequently ſimilar, by our 34th Propoſition; wherefore as the Sine EG to the Sine EI, ſo the Sine BH to the ſine BK, or as EG to BH ſo EI to BK, and contrariwiſe.

SCHOLIUM VI.

HEnce you have ſeveral Rules of *Spherical Trigonometry* for reſolving Right Angled ΔΔ (α) 1. Having given in the Rightangled Δ ABC the Hypothenuſe BC and the Oblique Angle ACB, for the Leg AB oppoſite to this Angle, make: as the ſine T (EG) to the ſine of the Hypoth. (BH) ſo the ſine of the given Angle (EI) to the ſine of the Leg ſought (BK). 2. Having given the Hypothe-

(α) *Lansberg. Geom. Triang. lib.* 4. *Prop.* 12.

nuſe BC and the Leg AB for the oppoſite Angle ACB make as the ſine Hypoth. (BH) to *S.* T. (EC) ſo the ſine of the given Leg (*B*K) to the ſine of the Angle ſought (EI) 3. Having given the ſide AB and the Angle oppoſite to it ACB, for the Hypothenuſe BC (ſuppoſing you know whether it be greater or leſs than a Quadrant) make as the ſine of the given Angle EI to the ſine T. (EG) ſo the ſine of the given Leg (*B*K) to the ſine of the Hypoth. BH). 4. Having given in the Right Angled △ EBF (which we take inſtead of ABC, that ſo we may not be obliged to change the Figure) one Leg EB and the Hypothenuſe BF for the other Leg *E*F, you may find its complement, if you make as the ſine Complement of the given ſide (BH) to *S.* T. (EG) ſo the ſine Complement of the Hypothenuſe (BK) to the ſine Compl. of the ſide ſought (EI) 5. Having both Legs EB and EF given, for the Hypothenuſe BF its Compl. BA may be found thus: as *S.* T. (EG) is to the ſine Cpmpl. (BH) of one ſide EB, ſo the ſine Compl. (EI) of the other ſide (EF) to (BK) the ſine Compl. of the Hypothenuſe.

6. Having given in the ſame Right Angled Triangle EBF one Leg EF, and the Angle adjacent to it ETB, firſt prolong into whole Quadrants BA to *f*, that A*f* may = BF Hypoth. & BC to *e* that C*e* may = EB, and AC to *d* that C*d* may = DA the meaſure of the given Angle EFB: ſecondly from *d* thro' *e* and *f* let fall a Quadrant thro' the extremities of the Quadrants B*f* and B*e*, that ſo the △ C *de* may be Right Angled, in which there are given the Hypoth. C*d* = to the given Angle, and the Angles C = to the Compl. of the given Leg (*viz.* to the Arch ED) and ſo, thirdly, there is ſought the ſide *de*, as the Complement of the Arch *ef*, or of the Angle ſought ABC, or EBF; *viz.* by the firſt caſe of this, by inferring, as *S.* T. to the ſine Hypoth. *cd* (*i.e.* of the given Angle EFB;) ſo the Angle *dce* (*i. e.* DE the Compl. of the given Leg RF) to the ſine *de* (as the Compl. of the Angle *f*B*e* or EBF.)

7. Having given, in the ſame Triangle, the ſide EF and the oppoſite Angle EBF (*i. e.* the Arch *ef*) for the other Angle EFB, (that is the Hypoth. *cd* in the △ *cde*) make by the third of this:

As the *Sine* of the Angle *dce* (*i. e.* the ſine Compl. of the given Leg DE) to the *S.* T. ſo the ſine of the Leg *de* (*i. e.* the ſine

ſine Compl. of the Angle EBF) to the Hypoth. *cd* (*i.e.* the ſine of the Arch DA or Angle EFB.).

8. Having the Oblique Angles given to find either of the ſides, *viz.* EF; which may be done thus by the ſecond of this: As the ſine of the Hypoth. *c d* (*i. e.* the ſine of the Angle at F) to the W. S. ſo the ſine *de* (*i. e.* the ſine Compl. of the Angle at B) to the ſine of the Angle *dc e* (*i.e.* the ſine Compl. of the ſide ſought EF.)

Conſect. 10. The ſame being given as in *Conſect.* 7. if inſtead of the Right Sines EI and BK, you erect Perpendicularly DL and AM(*Fig.*71) becauſe of the ſimilitude of the Triangles DGL and AHM, you'l have, as DG ſine T. to DL the Tangent of the Arch DE, ſo AH the Right Sine of the Arch AC to AM the Tangent of the Arch AB; or as DG to AH, ſo DL to AM, and contrariwiſe.

SCHOLIUM VII.

HEnce flow the other Rules of *Spherical Trigonometry* for Reſolving Right Angled Triangles, *viz.* 9. Having given the ſide AC in the △ABC, and the adjacent Angle ACB, for the other ſide AB, make as the W.S. (DG) to the ſine of the given ſide (AH) ſo the Tangent of the given Angle (ACB) to the Tangent of the Angle ſought (AB.) 10. Having given the ſide (AB) and the oppoſite Angle (at C) for the other ſide (AC, ſo you know whether it be greater or leſs than a Quadrant) make as the Tangent of the given Angle (DL) to the Tangent of the given Leg (AB) ſo the whole S. (DG) to the ſine of the Leg ſought (*viz.* at AH.) 11. Both ſides being given, for the Angles, make, as the ſine of one Leg (AH) to the W. *S.* (DG) ſo the T. of the other Leg (AM) to the Tangent of the Angle oppoſite to the ſame (at C.) 12. Having given moreover in the Right Angled Triangle EBF the Hypothenuſe (BF) and the Angle (EFB) for the adjacent ſide EF, make, as the ſine Compl. of the given Angle (AH) to the W. S. ſo the Tangent Compl. of the Hypoth. (AM) to the Tang. Compl. of the Leg ſought (DL) 13. Having given the ſide (EF) and the adjacent Angle F for the Hypoth. BF make; as the W. S. to the ſine Compl. of the given Angle (AH) ſo the Tangent Complement of the given Leg (DL)

to

to the Tangent Compl. of the Hypoth. (AM.) 14. Having given the Hypoth. (BF) and one side EF for the adjacent Angle (F) make as the Tang. Compl. of the given Leg (D.L.) to the W. S. so the Tang. Compl. of the Hypoth. (AM) to the Sine Compl. of the Angle sought (AH). 15. Having given the Hypoth. (BF, *i. e.* the arch Af, or the angle at *d*) and either of the oblique angles (at F) for the other angle (EBF) make by help of the new Triangle *c de*, by the 12*th* of this.

As the Sine Compl. of the angle *c d e* (*i. e.* the Sine Compl. of the Hypoth. (AH) to the W. S. so the Tang. Compl. of the Hypoth. *c d* (*i. e.* Tang. Compl. of the given angle) to the Tang. Compl. of the Side *d e* (*i. e.* to the Tang. of the angle sought ABC or EBF.)

16. Having given the oblique Angles to find the Hypoth. (BF, or the arch A*f*, or the angle *c d e*) it is done by the 14 of this *Schol.*

As the Tang. Compl. of the Leg *d e* (*i. e.* the Tangent of the angle ABC or EBF) to the W. S. so the Tangent Complement of the Hypoth. *c d* (*i. e.* the Tangent Complement of the other angle EFB) to the Sine Compl. of the angle *c de* (*i. e.* the Sine Compl. of the Hypoth. BC sought.)

So that now we have with *Lansbergius* (but much more compendiously) Scientifically Resolved all the Cases of Rigt-angled Triangles; the Resolution of Oblique-angled ones only now remaining.

Consect. 11. In Oblique-angled Spherical Triangles, as well as Right-angled ones, the Sines of the angles are directly proportional to the Sines of the opposite Sines. 1. Of the Right-angled ones this is evident from N° 3. *Schol.* 6. and from the 9th *Consect.* For as the Sine of the angle A (*Fig.* 72.) to the Sine of BD, so the W. S. (*i.e.* of the angle D) to the Sine of AB. 2. The same is immediately evident of an Oblique-angled Triangle ABC, resolved into 2 Right-angled ones. For,

The Sine of the angle C is to the Sine of BD as the sine of the angle D to the Sine of AB; and also,

The Sine of the angle C to the Sine of BD as the Sine of the angle D to the sine of BC, by the 1.

In

In each Proportionality the means are the Sines of BD and D; therefore the Rectangles of the Extremes of the Sines of AB into the Sine of A, and the Sine of BC into the Sine of C, will be equal among themselves, since the Rectangles of the same Means are equal, by *Prop.* 18. therefore by *Prop.* 19. as the Sine of A to the Sine of BC, so the Sine of C to the Sine of AB, Q.E.D.

SCHOLIUM VIII.

THe latter may appear of Oblique-angled Triangles after this way also; since the Sine of the angle A is to the Sine of BD as the Sine of the angle D to the sine of AB, call the first a, the second $e\,a$, the third b, the fourth $e\,b$; and because the Sine of the angle C (which we call c) is likewise to the Sine of BD (*i. e.* to $e\,a$) as the Sine of D (*i. e.* b) to the Sine of BC (which will consequently be $\frac{e\,a\,b}{c}$) it will be manifest, that the Sine of the angle

A is to the sine of BC as the sine of the angle C to the sine of *AB*,

i.e. as a to . . $\frac{eab}{c}$, *i.e.* . . c to . . eb.

by multiplying the Means and Extremes, whose Rectangles are on both sides $e\,a\,b$. Therefore as by the present and precedent *Consectary* 7, it is universally true, *That in any Triangle whether Right Lined or Spherical, Right-Angled or Oblique-angled, the Sides or their Sines, are to one another, as the Sines of their opposite Angles* (which therefore is commonly called a *Common Theorem*:) so also hence flow 2 new Rules of *Spherical Trigonometry* for Oblique-angled Triangles, like those we found in *Schol.* 4.

To find		I. *The other Angles.*
From 2 sides given of an angle opposite to one of them,	by inferring	As the sine of the side opposite to the given angle to the sine of the other side, so the sine of the given angle to the sine of the angle sought.

II. *The*

II. *The other Sides.*

From one side and the angles given,	by inferring	As the sine of the angle opposite to the given side to the sine of that side, so the sine of the angle opposite to the side sought to the sine of the side sought.

And thus we have reduced all the Cases and Rules of Spherical Trigonometry to their original Fountains (for from 2 Sides given and the interjacent Angle, or 2 Angles and their adjacent side, we may find the rest in Oblique-angled Triangles, by resolving them into 2 Right-angled ones; and so by the Rules we have deduc'd in *Schol.* 6 and 7) excepting two Cases, *viz.* when from 3 sides given, the Angles, or from 3 Angles the Sides are sought; to resolve which, we are supplied with Rules from the following

Cons. 12. In the given Oblique-angled Spherical Triangle ABC (*Fig.* 73.) whose Sides are unequal and each less than a Quadrant, having produced the sides AB and AC to the Quadrant AD and AE, and effected besides what the Figure directs, then will

The Arch DE be the Measure of the angle A, AF$=$AC, and so FB the difference of the Sides AB and AC.

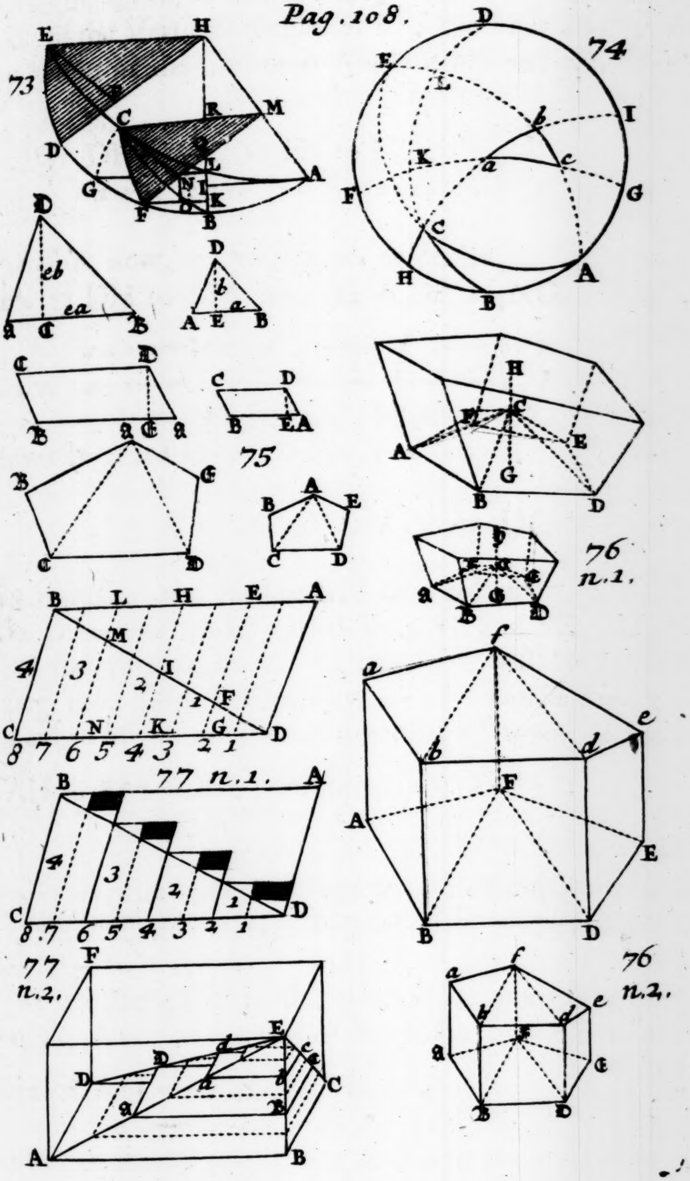
BC$=$BG, and so GF the difference of the third side, and the differences of the rest FB.

But now, 1. As EH or DH to CM or FM so will PH be to NM (by reason of the Equiangular Triangles EPH and CNM;) therefore by *Prop.* 26. so will also D P be to F N. Make therefore DH$=a$ FM$=$ ea, DP$=b$ FN$=eb$.

AI the R. Sine of the side AB.
GM the Sine of the Side AC.
GL the Sine of G*B*, or of the side BC.
FK the R. Sine of the Arch FB.
*B*I the versed Sine of A*B*.
*B*L the vers. Sine of G*B* or BC.
*B*K the versed *S*ine of F*B*.
KL or NO the difference of the versed Sines we have now mentioned.
EP the Right *S*ine and DP the versed *S*ine of the arch DE.
CN the R. *S*ine and FN the versed *S*ine of the arch FC.

2. By

Pag. 108.



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2. By reaſon of the Equiangular ΔΔ FNO and HAI (for FNO is Equiangular to the Δ FKQ, and that to the Δ HQM, by reaſon of the Vertical Angles at Q; and that alſo to the Δ HAI by reaſon of the Common Angle at H) and you have alſo,

As HA,
Or DH to AI, ſo FN to NO.
a — *oa* — *eb* — *oeb*.

Wherefore now, 3. you'l have evidently,
the □ DH to FM into AI as DP to NO.

	aa —	*oeaa* —	*b* —	*oeb*.
i. e.	*a* —	*oea* —	*b* —	*oeb*.
		DH	NO	DP.

And Inverſly as *oea* to *a* ſo *oeb* to *b*.

SCHOLIUM IX.

Since therefore the Radius DH or *a* is known, and alſo NO the Difference of the verſed Sines *BL* and *BK*, it is evident, that DP the verſed Sine of the angle A will be known alſo; ſuppoſing that the firſt Quantity *oea* is likewiſe known. But this may be had by another Antecedent Inference, if you make,

as AH to FM ſo AI to a fourth *oea*.
a *ea* *oa*

Hence therefore ariſes, 1. the Rule: Having given the 3 Sides of an Oblique-angled Triangle, to find any one of the Angles, *viz.* by inferring,

1. As the Sine of T to the Sine of R, one of the ſides comprehending AC; ſo the ſine of the other ſide AB to a fourth,

DH or AH — FM — AI—*oea*.
a — *ea* — *oa*.

2. As this fourth to the ſine of T, ſo the difference of the verſed ſines of the third ſide, BC, and the differences of the others to the verſed ſine of the Angle ſought, *viz.*

oea — *a* — NO — DP.
oeb *b*.

But ſince the ſides of a Spherical Triangle may be changed into

into Angles, and contrariwise the sides being continued [as the side AB of the given Triangle ABC (*Fig.* 47.) be continued a Circle, the rest into Semicircles from the Poles *b* and *c*, and likewise the Semicircle HI from the Pole A, and the Semicircle FG from the Pole B, and the Semicircle EA from the Pole C, you'l have a new Triangle *a*, *b*, *c*, the 3 angles of which will be equal to the 3 sides of the former ABC; as the angle *a* or its measure IG, is equal to the side AB, by reason each makes a Quadrant joined with the third arch AG; but the measure of the angle *b*, is the side AC (*viz.* in this case wherein the side AC is a Quadrant, in the other wherein it would be greater or less than a Quadrant, it would be the measure of the angle of the Compl. for then the Semicircle HI described from the Pole A, would not pass thro' C but beyond or on one side of C. See *Pitisc. lib.* 1. *Prop.* 61. *p. m.* 25.) ——— the angle *c* or its measure KL, is equal to the side BC, because with the third KC they make the Quadrants BK and CL] Therefore, 2. Having given the three Angles of the Oblique-angled Triangle *a b c*, you may find any side, *e. g.* *a c*, if there be sought the Angle ABC, or rather its Complement KBF, or its measure FK $=$ *a c*, ——— from the 3 sides given of the $\triangle$ ABC, by the preceding Rule, by inferring, *viz.* 1. As *S.* T. to the sine R, of one side comprehending the angle of one side AB (*i.e.* of one angle *a* adjacent to the side sought) so the sine of the other side BC (*i. e.* of the other angle C) to a fourth.

2. As the fourth to the *S.* T. so the difference of the versed Sines of the third side AC, and the differences of the others (*i. e.* of the 3d angle *b*, and the differences of the rest) to the versed Sine of the comprehended angle, or Complement to a Semicircle (*i. e.* of the side sought *a c.*)

Proposition XXXV.

***S**imilar Plane Figures (α) are to one another in Duplicate Proportion of their Homologous Sides.*

Demonstration.

For, 1. the Bases of 2 similar Triangles or Parallelograms

(For

(for which any 2 Homologous Sides, *e. g.* AB and $\mathfrak{AB}$ (*Fig.* 75.) may be taken) and Perpendiculars let fall thereon DE and $\mathfrak{DE}$, will be by *Consect.* 1. *Prop.* 34, as *a* to *e a*, *b* to *e b*. Therefore the Parallelograms and Triangles themselves, will be as *ba* to *eeba*, by *Consect.* 7 and 8. *Def.* 12. *i. e.* by *Def.* 34. in duplicate Reason of their Perpendiculars or assumed Sides, which is most conspicuous in Squares, which putting *a* for the side of one, and *e a* for the other, are to one another as *a a* to *e e a a*.

2. Like Polygons are resolved into like Triangles, when the Triangles ABC and $\mathfrak{ABC}$, and also AED and $\mathfrak{AED}$ are Equiangular, by *Consect.* 3. *Prop.* 34. but CAD and $\mathfrak{CAD}$, are also Equiangular, because each of their angles are the remainder of equal ones, after equal ones are taken from them. Wherefore the first Triangles are in duplicate Proportion of the sides BC and $\mathfrak{BC}$; the second likewise of the sides CD and $\mathfrak{CD}$ the third are also in the same Proportion of the sides DE and $\mathfrak{DE}$, *&c. i. e.* (since by the Hypoth. BC has the same reason to $\mathfrak{BC}$ as CD to $\mathfrak{CD}$, and DE to $\mathfrak{DE}$) each to each is in duplicate Proportion of the sides BC to $\mathfrak{BC}$, or CD to $\mathfrak{CD}$, by the first of this: Therefore by a *Syllepsis*, the whole Polygons are in duplicate Proportion of the same *Sides*: Which is the second thing to be demonstrated.

(α) *Eucl.* 19 & 20 *lib.* 6.

3. Circles and their like *Sectors*, are as the Squares of their Diameters, by *Prop.* 32. therefore in duplicate Proportion of them, by the first of this: Which is the third thing: Therefore similar Plane Figures, *&c.* Q. E. D.

CONSECTARYS.

I. THerefore 2 similar Plane Figures are one to another, as the first Homologous *Side*, to a third Proportional, by vertue of *Definition* 34.

II. Any two Figures described on 4 Proportional Lines (α) and similar to 2 others, are likewise Proportional, and contrariwise; for if the simple Reasons or Proportions of Lines be the same, their duplicate Proportions will be the same also, and reciprocally.

(α) *Eucl. prop.* 22. *lib.* 6.

SCHO-

SCHOLIUM.

BUT as this second Consectary confirms Prop. 22 and its Scholium, so the first teaches us a twofold Geometrical Praxis. 1. A Way to express the Proportion of similar Figures by two Right Lines, *viz.* by finding a third Proportional to their Homologous Sides. For as the side of the first to this third, so will be the first Figure to the second. 2. A way to augment or diminish any given Figure in a given Reason or Proportion, *viz.* by finding a mean Proportional between any side of the given Figure, and another Line which shall be to that in a given Proportion, and then by describing thereon a similar or like Figure.

Proposition XXXVI.

Similar or like Solid Figures, are to one another in triplicate Proportion of their Homologous Sides.

Demonstration.

For, 1. The similar Bases of two similar Parallelepipedons (and consequently also of Prisms and Cylinders, by *Consect.* ... and 5. *Definit.* 16. and also of Pyramids and Cones, *by Consect.* 3 and 4, of *Definit.* 17.) are, as ab to $eeab$, by (α) the preceding Proposition, and their Altitudes as c to ec, by *Consect.* ... *Prop.* 34.

Therefore Parallelepipeds, Cylinders and Prisms (and so the third part of these, Cones and Pyramids) will be as abc to e^3abc, by *Consect.* 3, 4, 5. *Definit.* 16. *i. e.* they will be, by *Definit.* 34. & *Consect.* 1 and 2. *Prop.* 34. in Triplicate Proportion of their Perpendiculars or Homologous Sides. Which is especially conspicuous in Cubes; which, putting a for the Side of one, and ea for the side, are to one another as a^3 to e^3a^3.

2. Polyedrous or many sided Figures, may be resolved into Pyramids of similar Bases and Altitudes; which is evident of Regular ones, from the *Consect.* of *Definit.* 21. and cannot be difficult to understand also of Irregular ones; because the like inclinatio...

Inclination of their Planes every where similar and equal in number, necessarily require that the whole Altitudes of similar Polyedrous Solids, as well as similar Parallelepipedons, by *Consect.* 2. *Prop.* 34. should be in subduplicate Proportion of their Bases, and so these being likewise divided in C and 𝕮 (*Fig.* 76. N° 1.) the parts of their heigths GC and 𝕲𝕮, will be in the same Proportion: Whence, *e. g.* 2 Pyramids standing on similar Bases ABDEF and 𝕬𝕭𝕯𝕰𝕱, and having like Altitudes GC and 𝕲𝕮, will necessarily be like or similar; and the same thing may be likewise judged of others.

Or yet, to shew it more evidently, the Polyedrous Solids may be resolved into like Triangular Prisms; for, *e.g.* each of the Triangles of their similar Bases ABDEF and 𝕬𝕭𝕯𝕰𝕱 (N° 2.) are similar, *viz.* 𝖆 𝖇 𝖋 and *a b f*, 𝕬𝕭𝕱 and ABF, by the preceding *Prop.* N° 2. The Planes 𝕬𝕭𝖇𝖆 and A*Bba*, also 𝕬𝖆𝖋𝕱 and A*af*F, are similar by the Hypoth. and consequently also the Planes 𝕭𝖇𝖋𝕱 and B*bf*F (𝖋𝖇 is to 𝖇𝖆 as *fb* to *ba*, and also in the one 𝖇𝖆 to 𝖇𝕭, as *ba* to *b*B in the other; therefore *ex equo* as 𝖋𝖇 to 𝖇𝕭 so *f b* to *b*B, &c.) and so the whole Triangular Prisms will be similar, by *Definit.* 35. and so of others. Therefore similar Polyedrous Solids will be in the same Proportion as similar Pyramids, or Triangular Prisms, *i. e.* by the first of this in Triplicate Proportion of their Sides.

(α) *Eucl. prop.* 12. *lib.* 12. *of Cones and Cylinders.*

3. Spheres are as the Cubes of their Diameters (β) by *Consect.* 3. *Prop.* 32. Therefore they are by the first of this, as a^3 to e^3a^3. Therefore similar or like Solids are in Triplicate Proportion of their Homologous Sides. Q. E. D.

(β) *Eucl.* 18. *lib.* 12.

CHAP. VI.

Of the Proportions of Magnitudes of divers sorts compared together.

Proposition XXXVII.

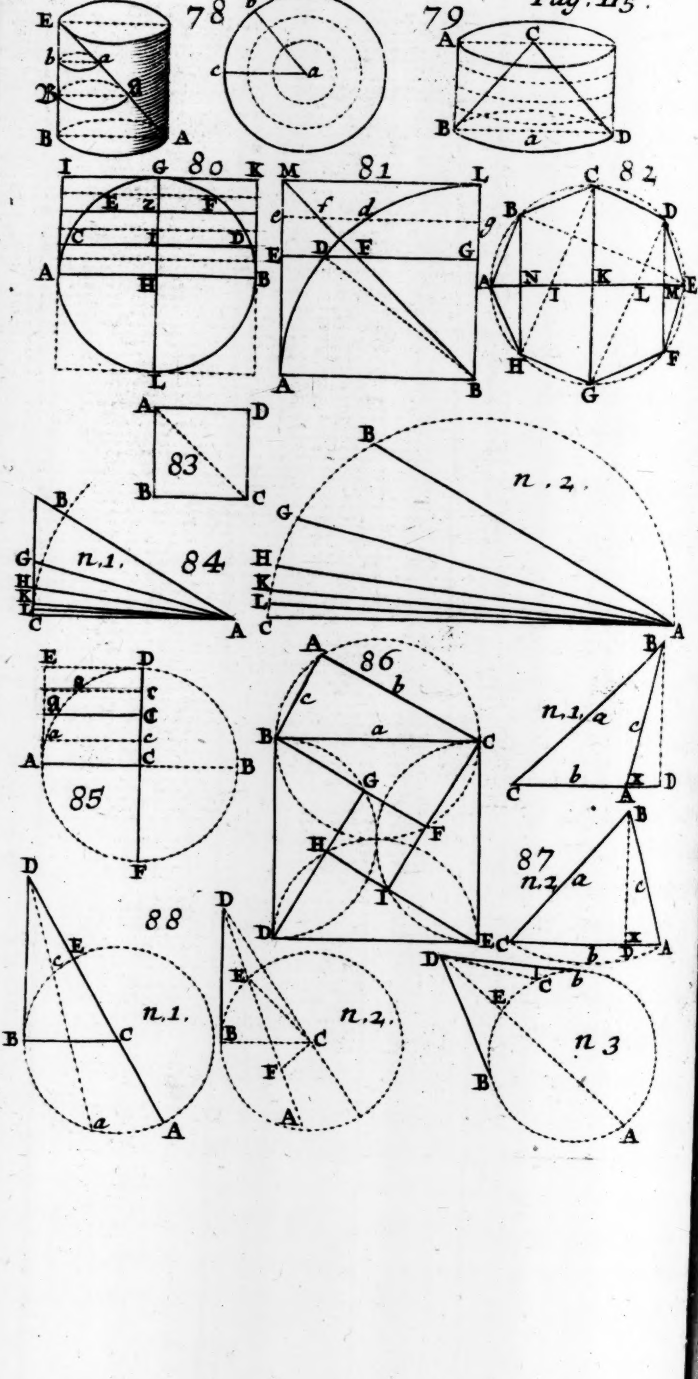
THE *Parallelogram ABCD* (Fig. 77. N. 1.) *is to the Triang BCD upon the same base DC, and of the same heighth as* 2 *to* 1. This has been already Demonstrated in *Consect.* 3. *Definit.* 12. Here we shall give you another

Demonstration.

Suppose, 1. the whole Base CD divided into four equal parts by the transverse Parallel Lines EG, HK, LN, then will (by reason of the similitude of the ΔΔ DGF, DKI, DNM DCB.) GF be 1, KI 2, NM 3, CB 4; and having furthermore continually Bisected the Parts of the Base, the Indivisibles or the Portions of the Lines drawn transversly thro' the Triangle will be 1, 2, 3, 4, 5, 6, 7, 8, *&c. ad infinitum*, all along in an Arithmetical Progression, beginning from the Point D, as 0; to which the like number of Indivisibles always answer in the Parallelogram equal to the greatest, *viz.* the Line BC. Wherefore by the 4th *Consect.* of *Prop.* 16. all the Indivisibles of the Triangle, to all those of the Parallelogram taken together, *i. e.* the Triangle it self to the Parallelogram, is as 1 to 2. Q. E. D.

SCHOLIUM.

NOW if any one should doubt whether the Triangle or Parallelogram may be rightly said to consist of an infinite number of Indivisible Lines, he may, with Dr. Wallis instead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instead of Lines, conceive infinitely little Parallelograms of the same infinitely little Heighth, and it will do as well. For having cut the Base (N° 2) into 4 equal parts by transverse Parallels, there will be circumscribed about the Triangle so many Parallelograms of equal heighth, being in the same Proportion as their Bases, by *Prop.* 28. *i. e.* increasing in Arithmetical Progression. In the following Bisection, there will arise 8 such Parallelograms approaching nearer to the Triangle, in the next 16, &c. so that at length infinite such Parallelograms of infinitely less heighth, and ending in the Triangle itself, will constitute or make an infinite Series of Arithmetical Proportionals, beginning not from 0 but 1; to which there will answer in the Parallelogram infinite little Parallelograms of the same heighth, equal to the greatest. Whence it again follows, by *Consect.* 9. *Prop.* 21. that the one Series is to the other, *i. e.* the Triangle to the Parallelogram as 1 to 2; which being here thus once explained, may be the more easily applied to Cases of the like nature hereafter.

CONSECTARYS.

I. Since in like manner in the Circle (*Fig.* 79.) the Peripherys at equal intervals from one another, as so many Elements of the Circle, increase in Arithmetical Progression; the Sum of these Elements, *i. e.* the Circle it self will be to the Sum of as many Terms equal to the greatest Periphery, *i. e.* to a Cylindrical Surface, whose Base is the greatest Periphery, and its Altitude the Semidiameter, as 1 to 2.

II. Hence the Curve Surface of a Cylinder circumscribed about a Sphere, *i. e.* whose Altitude is equal to the Diameter, is Quadruple to its Base.

III. Also the Sector of the Circle *b a c*, to a Cylindrical Surface, whose Base is the Arch *b c*, but its Altitude the Semidiameter *a c*, is as 1 to 2.

IV. And because the Surface of the Cone BCD is to its circular Base, as BC to CA, *i. e.* as the $\surd 2$ to 1. by *Schol. Prop.* 17. the Cylindrical Surface, the Conical Surface and the Cir-

cular one we have hitherto made use of will be as 2, $\sqrt{2}$ and 1, and consequently continually Proportional.

SCHOLIUM

ALL which may also abundantly appear this way, *viz.* by putting for the Diameter of the Circle *a*, for the Semidiameter $\frac{1}{2}a$ and for the Circumference *ea*, you'l have the Area of the Circle $\frac{1}{4}eaa$ by *Consect.* 1 *Definit.* 31. and Multiplying the heighth of the Cylinder AB *i.e.* $\frac{1}{2}a$ by the Periphery *ea* you'l have the Cylindrical Surface $\frac{1}{2}eaa$ by *Consect.* 6. *Definit.* 18 as now is evident also by *Consect.* 1, 2 and 3. Now if you would also have the Surface of the Cone, since it's side by the *Pythagorick Theorem* is $\sqrt{\frac{1}{2}aa}$ and the half of that $\frac{1}{2}\sqrt{\frac{1}{2}aa}$ *i.e.* (by N° 2 of *Schol. Prop.* 22.) $\sqrt{\frac{1}{8}aa}$; and this half being multiplied by the Periphery of the Base *ea*, you'l have (by virtue of *Consect.* 4. *Definit.* 18.) the Surface of the Cone $ea\sqrt{\frac{1}{8}aa}$ *i.e.* (by the *Schol.* just now cited) $\sqrt{\frac{1}{8}eeaa^4}$: So that now appears also the 4*th Consect.* of this; because the Rectangle of those Extremes *eaa* and $\frac{1}{4}eaa$ is $\frac{1}{8}eaa^4$ as well as the square of the mean.

Proposition XXXVIII.

A *Parallelepiped* (α) *BF* (Fig. 77. N° 3.) *is to a Pyramid ABCDE upon the same Base BD and of the same heighth, as* 3 *to* 1. This was Demonstrated in *Consect.* 3. *Definit.* 17. but here we shall give you another.

Demonstration.

Suppose 1 the whole Altitude BE divided into 3 equal Parts by transverse Plains Parallel to the Base, then will (by reason of the Similitude of the Pyramids abcd E 𝔄𝔅ℭ𝔇E and ABCDE) the Bases abcd 𝔄𝔅ℭ𝔇 and ABCD be by *Consect.* 2. *Prop.* 34 and *Consect.* 3. *Definit.* 17 in duplicate Proportion of the Altitudes *i. e.* in duplicate Arithmetical Progression 1, 4, 9, moreover 2, bisecting the parts of the Altitude, the quadrangular Sections now double in Number (as the Indivisibles or Elements of the proposed Pyramid) will be as 1, 4, 9, 16, 25,

(α) *Euclid. Prop* 7 *Coroll. lib.* 12.

36

36, *&c. ad Infinitum*, all along in a duplicate Arithmetical Proportion; while in the mean time there anſwer to them as many Elements in the Parallelepiped equal to the greateſt ABCD, wherefore by *Conſect.* 10. *Prop.* 21. all the Indiviſibles of the Pyramid taken together will be to all the Indiviſibles of the Parallelepiped alſo taken together, *i e.* the Pyramid it ſelf to the Parallelepiped, as 1 to 3. Q. E. D.

CONSECTARY.

THis Demonſtration may be eaſily accommodated to all other Pyramids and Priſms, and alſo Cones and Cylinders, (*α*) ſince here alſo (*Fig.* 78.) the circular Planes *ba*, 𝔅𝔄, and BA are as the ſquares of the Diameters, and ſo as 1, 4, 9. and ſo likewiſe all the other Elements of the Cone by continual biſection are in duplicate Arithmetical Progreſſion; when in the mean time there anſwer to them in the Cylinder as many Elements equal to the greateſt BA, *&c.*

(*α*) *Euclid. Prop.* 10. *lib.* 12.

Propoſition XXXIX.

A *Cylinder is to a Sphere inſcribed in it* i. e. *of the ſame Baſe and Altitude as* 3 *to* 2.

Demonſtration.

Suppoſe 1 (*Fig.* 80) the half Altitude GH (for the ſame proportion which will hold when demonſtrated of the half Cylinder AK and Hemiſphere AGB, will alſo hold the ſame of the whole Cylinder to the whole Sphere) to be divided into 3 equal parts, then will AH, C1, E 2, be mean proportionals between the Segments of the Diameter by *Prop.* 34 *Schol.* 2 N° 3, and ſo by *Prop.* 17. the Rectangles LHG, L1 G, L2G equal to the Squares AH, C1, E2, being in order as 9 8 and 5. and alſo 2*dly*, having biſected the former parts of the heighth, the ſix Squares cutting the Sphere Croſs ways will be found to be as 36, 35, 32, 27, 20, 11. *&c.* in the progreſſion we have ſhewn at large in *Conſect.* 12. *Prop* 21. Wherefore ſince all the Indiviſibles of the Hemiſphere, *viz.* the circular Planes anſwering to the Squares of the ſaid tranſverſe

Diameters have the same proportion of Progression, by *Prop.* 32. and there answer to them the like number of Elements in the Cylinder equal to the greatest AH: All these taken together will be to all the other taken together *i. e*, the whole Cylinder AK to the whole Hemisphere AGB by vertue of the aforesaid *Consect.* 12. as 3 to 2 (*α*) Q. E. D.

(*α*) *Archim.* 32 (*al.* 31.) *lib.* 1 *de Sphær.*

SCHOLIUM. I.

Honoratus *Fabri* elegantly deduces this *Prop. a priori*, in a genetick Method in his *Synopsis Geom.* p. 318. (which also *Carolus Renaldinus* performs from the same common Foundation in *lib.* 1. *de Compos.* and *Resol.* p. 301, and the following, but after a more obscure way and from a demonstration further fetch'd) *Fabri's* is after this Method: The whole Figure (81) AL being turned round about BZ, the Quadrant ADLBA will describe a Hemisphere, the Square AZ a Cylinder and the triangle BML a Cone all of the same Base and Altitude. Since therefore Circles are as the squares of their Diameters by *Prop.* 32. and the Square of GE = to the Squares of GD and GF taken together (for the Square of GF *i. e.* GB × □ GD is = □ BD or BA or GE by the *Pythag. Theor.*) and so the Circle described by GE will be = to 2 Circles described by GD and GF taken together, then taking away the Common Circle described by GF there will remain the circle described by GF within the Cone equal to the *Annulus* or Ring described by DE about the Sphere. And since this may be demonstrated after the same way in any other case, *viz.* that a circle described by *g f*, will be equal to an *Annulus* described by *d e*; it will follow, that all Rings or *Annuli* described by the Lines DE or *de* (*i. e.* all that Solid that is conceived to be described by the trilinear Figure ADLM turned round) will be equal to all the Circles described by GF or *gf* (*i. e.* to the Cone generated by the Triangle BLM;) and so as the Cone is $\frac{1}{3}$ part of the Cylinder generated by AL, by the *Consect.* of *Prop.* 38. so also the Solid made by the Trilinear ADLM (*viz.* the Excess of the Cylinder above the Sphere) will be $\frac{1}{3}$ of the Cylinder, and consequently the Hemisphere $\frac{2}{3}$ Q. E. D.

CON

CONSECTAYS.

HEnce you have a further Confirmation of *Conſect.* 2. *Prop.* 32. and *Prop.* 36. N. 3.

II. Hence alſo naturally flows a Confirmation of *Conſect.* 2. *Definit.* 20. and conſequently the Dimenſion of the Sphere both as to its ſolidity and Surface. For putting *a* for the Diameter of the Sphere and circumſcribed Cylinder, and E*a* for the Circumference, the Baſis of the greateſt Circle will be $\frac{1}{4}eaa$, and that multiplied by the Altitude, gives $\frac{1}{4}ea^3$ for the Cylinder. Therefore by the preſent Propoſition, $\frac{1}{6}ea^3$ gives the Solidity of the Sphere (by making as 3 to 2 ſo $\frac{1}{4}$ to $\frac{1}{6}$) This divided by $\frac{1}{6}a$, will give, by vertue of *Conſect.* 1. of the aforeſaid *Def.* 20. and *Conſect.* 3. *Definit.* 17. the Surface of the Sphere *eaa*.

III. Therefore the (α) Surface of the Sphere *eaa*, is manifeſtly Quadruple of the greateſt Circle $\frac{1}{4}eaa$.

IV. The Surface of the Cylinder, without the Baſes, made by multiplying the Altitude *a* by the Circular Periphery of the Baſe *ea*, will be *eea*, equal to the Surface of the Sphere.

V. Adding therefore the 2 Baſes, each whereof is $\frac{1}{4}eaa$, the whole Surface of the Cylinder $1\frac{1}{2}eaa$, will be to the Surface of the Sphere *eaa* as 3 to 2.

VI. The Square of the Diameter *aa* to the Area of the Circle $\frac{1}{4}eaa$, is as *a* to $\frac{1}{4}ea$, *i. e.* as the Diameter to the 4th part of the circumference.

VII. A Cone of the ſame Baſe and Altitude with the Sphere and Cylinder, will be by *Conſect.* 2. of this, *Prop.* and the *Conſect.* of *Prop.* 38. $\frac{1}{12}ea^3$, and of the Cylinders $\frac{1}{4}$ or $\frac{3}{12}ea^3$. Therefore a Cone, Sphere, and Cylinder, of the ſame heighth and diameter, are as 1, 2, 3. The Cone therefore is equal to the Exceſs of the Cylinder above the Sphere; as is otherwiſe evident in *Scholium* 1. of this.

(α) *Archim. lib.* 1. *de Sph. & Cylind. Prop.* 31. (al. 30).

SCHOLIUM II.

AND thus we have briefly and directly demonstrated the chief Propositions of *Archimedes*, in his 1. Book *de Sphær. & Cylind.* which he has deduced by a tedious *Apparatus*, and only indirectly. And now if you have a mind to Survey the longer and more perplext way of *Archimedes*, and compare it with this shorter cut we have given you; take it thus: *Archimedes* thought it necessary first of all to premise this *Lemma*; That all the Conical Surfaces of the Conical Body made by Circumvolution of the Polygon, or many-angled Figure A,B,C,D,E, *&c.* (*Fig.* 82.) inscribed in a Circle, according to *Definit.* 19 I say, those Conical Surfaces taken all together, will be equal to a Circle, whose Radius is a mean Proportional between the Diameter AE and a transverse Line BE, drawn from one extremity of the Diameter E to the end of the side AB next to the other extremity. This we will thus demonstrate by the help of specious Arithmetick: Since BN, HN are the Right Sines of equal Arches, CK and CK whole Sines, *&c.* and the Lines BH, GC, *&c.* parallel; having drawn obliquely the transverse Lines HC, GD, all the angles at H, C, G, D, *&c.* will be equal by *Consect.* 1. *Definit.* 11. and consequently all the Triangles BNA, HNI, ICK, *&c.* equiangular, both among themselves, and to the △ABE; since the angle at B is a Right one, by *Consect.* 1 *Prop.* 33. and the angle at A common with the △BNA. Wherefore as BN to NA so CK to CI.

or HN to NI & GK to KL.

and so DM to ML FM to ME so EB to BA; and so

by making BN, HN, DM, FM $= a$ CK and GK $= b$, EB $= c$, for NA, NI, ML and ME, you may rightly put ea for KI and KL eb for AB, ec. Which being done you may easily obtain the Conical Surfaces of the inscrib'd Solid, and the Area of a Circle whose Radius shall be a mean Proportional between AE and FB, and it will be evidently manifest, that these two are equal. For, 1. (*for Conical Surfaces*) the Diameter of the *Base* $BH = 2a$, and the side of the Cone $AB = ec$: Therefore (making here o the name of the Reason between the Diameter and Circumference) the circumference will be $2oa$

which

which multiplied by half the ſide $\frac{1}{2}ec$, gives the Conical Surface $oaec$, by *Conſect.* 4. *Definit.* 18. And ſince the Circumference *B*H is as before $2\,oa$, and the circumference CG $= 2\,ob$, half of the ſum $2\,oa + 2\,ob$, *viz.* $oa + ob$ is the *equated* Circumference: which multiplied by the Side *B*C $= ec$ gives the Surface of the truncated Cone *B*HGC $= oaec - obec$, by *Conſect* 5. of the aforeſaid *Def.* ſince, laſtly, the Surface of the truncated Cone DFGC is equal to one, and likewiſe the conical *S*urface EDF to the other, by adding you'l have the Sum of all $4oaec + 2obec$.

2. (for the Area of the Circle) the Diameter AE is $= 4ea + 2eb$, and *B*E $= c$: the Rectangle of theſe is $= 4eac + 2ebc =$ (which alſo is evidently equal to the Rectangle of all the tranſverſe Lines *B*H, CG, DF into the ſide A*B*, as *Archimedes* propoſes in the matter) $=$ to the Square of the Radius in the Circle ſought, becauſe the Radius is a mean Proportional between AE and *B*E, and ſo equal to $\sqrt{4eac + 2ebc}$, ſo that the whole Diameter is $2\sqrt{4eac + 2ebc}$. Therefore, 2. the circumference of this Circle will be $20\sqrt{4eac + 2ebc}$, i. e. $\sqrt{1600eac + 800ebc}$: which multiplyed by half the Semidiameter, *i. e.* by $\frac{1}{2}\sqrt{4eac + 2ebc}$, *i. e.* $\sqrt{eac + \frac{1}{2}ebc}$ gives the Area of the Circle ſought $\sqrt{1600aaeecc + 1600eeabcc + 400eebbcc}$. But this Root extracted is $40aec + 20bec$, equal to the ſuperiour Sum of the conical Surfaces. Q.E D.

Having thus demonſtrated the Lemma, we will eaſily demonſtrate with *Archimedes* (tho not after his way) That the Surface of any *S*phere, is Quadruple of the greateſt Circle in it, which is already evident from the 3d *Conſect.* For ſince all the conical Surfaces of the inſcribed Solid taken together, by the preceding Lemma, are equal to the Area of a Circle whoſe Radius is a mean Proportional between the Diameter AE and the Tranſverſe E*B*; and this mean Proportional approaches always ſo much nearer to the *D*iameter AE, and thoſe Surfaces ſo much nearer to the Surface of the Sphere, by how many the more ſides the inſcribed Figure is conceived to have, by *Conſect.* 1 and 2. *Def.* 18. if you conceive in your mind the *B*iſection of the Arches A*B*, *B*C, *&c.* to be continued *in Infinitum*, it will neceſſarily follow, that all thoſe conical *S*urfaces will at length end in the *S*urface of the *S*phere it ſelf, and that mean Pro-

portional in the diameter AE, and so the Surface of the Sphere will be equal to a Circle, whose Radius is the Diameter AE. But that Circle would be Quadruple of the greatest Circle in the present Sphere, by *Prop.* 35. Therefore the Surface of the Sphere is Quadruple of that Circle also. Q. E. D.

Hence also it would be very easie to deduce with *Archimedes* (tho again after another way) that celebrated Proposition, which we have already demonstrated from another Principle in the Prop. of this Schol. *viz.* That a Cylinder is to a Sphere of the same Diameter and Altitude, as 3 to 2. For by putting a for the Diameter and Altitude, and $e\,a$ for the Circumference, the Area of the Circle, will be $\frac{1}{4}e\,a\,a$; and this Area being multiplied by the Altitude a, gives $\frac{1}{4}\,e\,a^3$ for the Cylinder, by *Consect.* 5. *Definit.* 16. and the same Quadruple, *i. e.* $e\,a\,a$ multiplied by $\frac{1}{6}a$ gives $\frac{1}{6}e\,a^3$ for the Sphere, by *Consect.* 1. *Definit.* 20. and *Consect.* 3. *Definit.* 17. Wherefore the Cylinder will be to the Sphere as $\frac{1}{4}$ to $\frac{1}{6}$, *i. e.* in the same Denominator as $\frac{6}{24}$ to $\frac{4}{24}$, *i. e.* as 6 to 4, or 3 to 2. Q.E.D.

Whence it is evident, that the Dimension of the Sphere would be every ways absolute if the Proportion of the Diameter to the Circumference were known; which now with *Archimedes* we will endeavour to Investigate.

Proposition XL.

THE *Proportion of the Periphery of a Circle* (α) *to the Diameter, is less than* $3\frac{1}{7}$ *or* $\frac{10}{70}$ to 1. and greater than $3\frac{1}{71}$ to 1.

Demonstration.

The whole force of this Proposition consists in these, that, 1. Any Figure circumscribed about a Circle, has a greater Periphery than the Circle, but any inscrib'd one a less. 2. The Periphery of a circumscribed Figure of 96 sides, has a less Proportion to the Diameter, than $3\frac{1}{7}$ to 1.

To demonstrate this second, we will enquire 3. the Proportion of one side of such a Figure, whether circumscribed or inscribed after the following way.

(α) *Archim. Cyclom. prop.* 2

For

For the first part of the Proposition.

Suppose the Arch *BC* (*Fig.* 84. N. 1.) of 30 degrees, and its Tangent *BC* making with the Radius AC a Right Angle, to make the Triangle ABC half of an Equilateral one, so that AB shall be to BC in double Reason, *viz.* as 1000 to 500; which being supposed, AC will be the Root of the Difference of the Squares BC and AB, *i. e.* a little greater than 866, but not quite $\frac{3}{100}$.

Then continually Bisecting the Angles BAC by AG, GAC by AH, HAC by AK, KAC by AL, BC is half the side of a circumscribed Hexagon, GC the half side of a Dodecagon (or 12 sided Figure) HC of a Polygon of 24 sides, KC of one of 48; lastly, LC of one of 96 sides; and by N. 3. *Schol.* 3. *Prop.* 34. GC will be to AC as BC to BA + AC, and also HC to AC as GC to GA + AC, *&c.* Wherefore

In the first Bisection, of what parts GC is 500, of the same will AC be 1866 and a little more, and AG (which is the Root of the Sum of the □ □ GC and AC) $1931\frac{8}{10}$ +.

In the second Bisection, of what parts HC is 500, of the same will AC be found to be $3797\frac{8}{10}$ + and AH $3830\frac{5}{10}$ +.

In the third Bisection, of what parts KL is 500 of the same will AC be $7628\frac{3}{10}$ + and AK $7644\frac{6}{10}$ +.

In the 4th Bisection, of what parts LC is 500 of the same will AC be $15272\frac{9}{10}$ +.

Now therefore LC taken 96 times, will give 48000 the Semi-periphery of the Polygon, which has the same Proportion to the Semi-diameter AC $15272\frac{9}{10}$ as the whole Periphery to the whole Diameter. But 48000 contains $15272\frac{9}{10}$, 3 times, and moreover $2181\frac{3}{10}$ remaining parts, which are less than $\frac{1}{7}$ part of the division, for multiplied by 7 they give only $1526\frac{1}{10}$ +.

Therefore it is evident, that the Periphery of this Polygon (and much more the Periphery of a less Circle than that) will have a less Proportion to the Diameter, than $3\frac{1}{7}$ to 1. Which is one thing we were to demonstrate.

For

For the 2d *Part of the* Prop.

SUppose the Arch BC to be (N° 2.) of 60 Degr. that is the Angle BAC at the Periphery of 30 *S*ince the Angle at B is a right one by *Conseᴄt*. 1. *Prop*. 33. the Triangle ABC will be again half an Equilateral one, and BC the whole side of an Hexagon, and GC of a Dodecagon, &c. *S*o that putting for BC 1000 (as before we put 500 for the side of the Hexagon) let AC be 2000 and AB the Root of the difference of the Squares BC and AC *i. e.* less than $1732\frac{1}{10}$ *viz*. 1732 and not quite $\frac{6}{100}$.

Then bisecting continually the Angles BAC, GAC, &c. since the Angles at the Periphery BAG, GAC, GCB, standing on equal Arches BG and GC, are equal by *Prop*. 33. and the Angle at C, (common to the Triangles GCF and GCA) and the others at H, K, L are all right ones by *Conseᴄt*. 1. of the aforesaid *Prop*. these 2 Triangles CGF and CGA are equiangular and consequently by *Prop*. 34. the Perpendicular GC in the one will be to the Perpendicular GA in the other as the Hypothenuse CF in the one to the Hypoth. AC in the other *i. e.* (by the foundation we have laid in the former part of the Demonstration of N° 3, *Schol*. 3. *Prop*. 34.) as BC to AB+AC ; and in like manner in the following HC will be to HA as GC to AG+AC, &c. Wherefore.

In the first Bisection, of what parts GC is 1000 of the same AG will be a little less then $3732\frac{1}{10}$; and AC (which is the Root of the Sum of the □□ AG and GC) will be a little less then $3863\frac{8}{10}$.

In the second Bisection, of what parts GC is 1000 of the same will AH be a little less then $7595\frac{9}{10}$ and AC a little less then $766\frac{5}{10}$.

In the third Bisection, of what parts KC is 1000 of the same will AK be a little less then $15257\frac{4}{10}$ and AC a little less $15290\frac{7}{10}$.

In the fourth Bisection, of what parts LC is 1000 of the same will AL be a little less then $30547\frac{6}{30}$; and AC a little less then 30564, and consequently if LC be put 500, AC will be less then 15282.

Now

Now therefore LC taken 96 times will give 48000 for the Periphery of the inscrib'd Polygon, and 15282 and a little less for the Diameter AC. But 48000 contains 15282 thrice, and moreover a remainder of 2154 parts, which are more then $\frac{10}{71}$ of the Divisor; for $\frac{1}{71}$ of this number makes $215\frac{17}{71}$ and so $\frac{10}{71}$ makes $2150\frac{170}{71}$ *i. e.* $2152\frac{28}{71}$. Therefore it is evident that the Periphery of this Inscribed Polygon (and much more the Periphery of a Circle greater then that) will be to it's Diameter in a greater proportion then $3\frac{1}{71}$ to 1, which is the 2*d* thing.

SCHOLIUM

If any one had rather make use of the small numbers of *Archimedes*, which he chose for this purpose, by putting in the first part of the Demonstration, for AB 306 and for BC 153, in the second for AC 1560 and for BC 780, by the like process of Demonstration, he may infer the same with *Archimedes*. We like our Numbers best, tho' somewhat large, because they may be remember'd, and are more proportionate to things, and make also the latter part of our Demonstration like the former. The Proportion in the mean while of the Diameter to the Periphery of the Circle by the *Archimedean* way is included within such narrow Limits, that they only differ from one another $\frac{1}{497}$ or $\frac{10}{4970}$ parts; for $\frac{10}{71}$ Subtracted from $\frac{10}{70}$ leave $\frac{1}{497}$, as $3\frac{10}{71}$ or $\frac{223}{71}$ and $3\frac{16}{70}$ or $\frac{220}{70}$ if they are reduced to the same Denomination make on the one hand $\frac{15610}{4970}$ on the other $\frac{15620}{4970}$. Hence it would be easy, having divided the difference $\frac{10}{4970}$ into 2 Parts, to express a middle proportion of the Periphery to the Diameter between the 2 Archimedean and Extreme ones as in these Numbers, 1561 to 4970, (or by dividing both sides by 5) as 3123 to 994, or (dividing both sides by 7) as $446\frac{1}{7}$ to 142, or (by dividing again by 2) as $223\frac{1}{14}$ to 71, &c.

While these Numbers become as fit for use as those of *Archimedes*, which we therefore use before any other, particularly in Dimensions that dont require an exact Niceness; where they do those may be made use of exhibited by *Ptolomey Vieta*, *Ludolphus a Ceulea*, *Metius*, *Snellius*, *Lansbergius*, *Hugeus*, &c. as if

The Diameter be . . . . The Circumference will be
10, 000, 000 . . . . 31, 416, 666. *Ptolomey.*
10 000, 000, 000 . . . . 31, 415, 926, 535. *Vieta.*
100, 000, 000, 000, 000, 000 000 . . . . 314, 159, 265, 358, 979, 323, $846\frac{1}{2}$, &c. *Ludolph. a Ceulen.*

Proposition XLI.

THe Area of a Circle has the same proportion to the Square of its Diameter, as the 4th *part of the Circumference to the Diameter.*

Demonstration.

Though we have before Demonstrated this Truth in *Consect.* 6 *Prop.* 39. yet here we will give it you again after another way. Since therefore the Circumference is a little less then $3\frac{1}{7}$, and a little more than $3\frac{10}{71}$ Diameters, for this excess putting z if the Diameter be 1 we will call the Circumference $3+z$ therefore the 4*th* Part of it will be $\frac{3+z}{4}$: And the Area of the Circle (by Multiplying the half Semidiameter) *i. e.* $\frac{1}{4}$ by the Circumference, you'l have both $\frac{3+z}{4}$ and the square of the Diameter = 1. Q. E. D.

CONSECTARY.

THerefore if the (α) Proportion of *Archimedes* be near enough truth to be made use of, *viz.* 22 to 7; the Area of the Circle will be to the Square of the Diameter as 11 to 14, because the quarter part of 22, *i. e.* $5\frac{1}{2}$ or $\frac{11}{2}$ to the Diam. 7 $\frac{14}{2}$ is in the same Proportion.

Proposition XLII.

THE Diameter (β) of a Square AC (*Fig.* 83.) is incommensurable to the side AB (and consequently also to the whole Periphery) *i. e.* it bears a Proportion to it that cannot be exactly expressed by Numbers.

(α) *Archim. Prop.* 3. *Cyclom.*
(β) *Eucl. last Prop. lib.* 10.

Demonstration.

For, if for AB you put 1, BC will be al-

1, and by the *Pythagorick Theorem* AC will be $=\sqrt{2}$. Therefore by *Consect.* 4. *Definit.* 30. AC is incommensurable to the side AB, *&c.* Q.E.D.

CONSECTARY I.

IT is notwithstanding Commensurable in Power; for its Square is to the Sqnare of the Side as 2 to 1.

CONSECTARY II.

NOW if the Proportion of the side or whole Periphery ABCDA, to the Diam. AC is to be expressed by Numbers somewhat near, as we have done in the Diameter and Circumference of a Circle; then making the side AB=100, the Diameter is greater than $141\frac{42}{100}$, less than $141\frac{43}{100}$.

Proposition XLIII.

THE Area of a Circle is Incommensurable to the Square of the Diameter.

Demonstration.

For dividing the Semidiameter CD (*Fig.* 85.) into two equal Parts (and consequently the Diameter DF into 4) AC will be 2, *viz.* $\sqrt{4}$ and AC $\sqrt{3}$, by *Schol.* 2. *Prop.* 34. N. 3. the sum $\sqrt{4}+\sqrt{3}$, and the sum of as many equal to the greatest AC 4. Having moreover Bisected the Parts of the Semidiameter, AC will $=4$ or the $\sqrt{16}$, *ac* $=\sqrt{15}$, AC $=\sqrt{12}$, AC $=\sqrt{7}$; the sum $\sqrt{16}+\sqrt{15}+\sqrt{12}+\sqrt{7}$; and the sum of as many equal to the greatest AC is $=16$, *&c.* And thus the last sums will be the Square Numbers increasing in Quadruple Proportion; but the former Sums will be always composed of the Rational Root of every such Square, and of several other irrational Roots of Numbers unevenly decreasing; so that it will be impossible to express those former Sums by any Rational Number, by what we have said in *Schol.* 2. *Definit.* 30. Wherefore all the Indivisibles of the Quadrant ADC are to as many of the Square ACDE equal to the greatest, *i. e.* the Quadrant it self

ADC

ADC to the Square ACDE (and consequently the whole Area of the Circle to this circumscrib'd Square) will be as a Surd Quantity to a true and truly Square Number, *i. e.* the Area of the Circle will be Incommensurable to the Square of the Diameter, by *Consect.* 4. of the said *Definit.* Q. E. D.

CONSECTARY.

AND because the fourth part of the Circumference has the same Proportion to the Diameter, as the Area of the Circle to the Square of the Diameter, by Prop. 41. Therefore also that will be Incommensurable to this, and consequently the whole Circumference will be so to the Diameter.

SCHOLIUM.

WHerefore it is somewhat Wonderful, which G. G. *Leibnitius* (α) tells us, that the Square of the Diameter being 1, the Area of the Circle will be $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}$, *&c. ad Infinitum. i. e.* by adding $1\frac{1}{5}-\frac{1}{3}$ and $+\frac{1}{9}-\frac{1}{7}$, &c.) to $\frac{2}{3}+\frac{2}{35}+\frac{2}{99}$, *&c. i. e.* to the Sum of infinite Fractions whose common Numerator is 2. But their Denominators Squares lessen'd by Unity, and taken out of the Series of the Squares of Natural Numbers by every fourth, omitting the Intermediate ones: Which Sum might seem expressible in Numbers, since all its parts are Fractions reducible to a common Denominaton; while notwithstanding *Leibnitius* himself confesses, that the Circle is not Commensurable to the Square, nor expressible by any Number.

(α) *Acta Erudit. Ann.* 82. *p.* 44 *& the following.*

CHAP. VII

CHAP. VII.

Of the Powers of the Sides of Triangles, and other Regular Figures, &c.

Proposition XLIV.

IN *Right-angled Triangles* (*a*) (*ABC*, Fig. 86.) *the Square of the Side* (*BC*) *that subtends the Right-angle, is equal to the Squares of the other Sides* (*AB and AC*) *taken together.*

Demonstration.

Though we have demonstrated this Truth more than once in the foregoing Proposition; yet here we will confirm it again as follows. Having described on each side of the Square BE a Semicircle, which will all necessarily touch one another in one point, and be equal to the Semicircle, BAC, if you conceive as many Triangles inscribed also equal to BAC; it will be evident that the Square BE will contain the said 4 Triangles; and besides the little Square FGHI, whose side FI, *v. g.* is the difference between the greater side of the Triangle CI, and the less CF, (for because the less side CF = BA, lying in the first Semicircle, if it be continued to I in the second Semicircle makes CI = CA the greater side of the other Triangle, and so in the others. From thence it is evident, That as the Angles ABC, and ACB together make one right one; so likewise BCF (= CBA) and ECF make also one right one; and consequently ECF is = ACB, and the Arch and the Line EI = to the Arch and the Line AB, &c.) Wherefore, if the greatest side of the given Triangle BC or BD, &c. be called *a*, and AC, *b* and the least AC, or CF, &c. be called *c*; the □ of the side BC, will be $= aa$, and the Area of each Triangle $\frac{1}{2}\,bc$: and so the 4 Triangles together $2\,bc$: but the side of the middle little Square will be $b - c$, and its Square $bb + cc - 2\,bc$: Wherefore if you add to this the 4 Trian-

(*a*) *Euc.* 47. *Lib.* 1.

The Sum of all, *i.e.* the whole Square BE will be $bb +$ $cc = aa$. Q. E. D.

CONSECTARYS.

I. Hence having the sides that comprehend the Right-ang[le] given, $AC = b$ and $AB = c$, the Hypothenuse or Ba[se] that subtends the Right-angle BC will be $= \sqrt{bb + cc}$.

II. But if BC be given $= a$ and $AC = b$, and you are [to] find $AB = x$; because $xx + bb = aa$; you'l have (taking aw[ay] from both sides bb) $xx = aa - bb$: therefore x, *i. e.* AB $=$ $\sqrt{aa - bb}$.

III. If 2 Right-angled Triangles have their Hypothenus[es] and one Leg equal, the other will also be equal.

Proposition XLV.

In Obtuse-angled Triangles (*Fig.* 87. *N.* 1. *the Square of the Ba[se] or greatest Side BC that subtends the Obtuse-angle BAC, is equ[al] to the Squares of* (α) *the other 2 Sides* (*AB and AC*) *taken togethe[r] and also to 2 Rectangles* (*CAD*) *made by one of t[he] Sides which contain the Obtuse-angle* (*AC*) *and t[he] continuation AD to the Perpendicular BD let fall fro[m] the other side.*

(α) Eucl. prop. 12. lib. 2.

Demonstration.

If BC be called a, $AB = c$, $AC = b$, $AD = x$, CD will b[e] $= b + x$. Therefore □ $BD = cc - xx$, by *Consect.* 2. of the pr[e]ceding *Prop.* In like manner if □ $CD = bb + 2bx + xx$ be su[b]tracted from the □ $BC = aa$, you'l have $aa - bb - 2bx - xx$ [=] to the same □ BD. Therefore

$$cc - xx = aa - bb - 2bx - xx,$$

i. e. (adding on both sides xx)

$$cc + aa - bb - 2bx.$$

i. e. (adding on both sides bb and $2bx$)

$$cc + bb + 2bx = aa. \quad \text{Q. E. D.}$$

CON

CONSECTARY.

IF in this last Equation you subtract from both Sides $cc+bb$ then will $2bx=aa-bb-cc$ and (if you moreover divide both Sides by $2b$) you'l have $x=\frac{aa-bb-cc}{2b}$: Which is the Rule, when you have the Sides of an Obtuse-angled Triangle given, to find the Segment AD, and consequently the Perpendicular BD.

Proposition XLVI.

IN *Acute-angled Triangles* (*a*) *the Square of any side* (e.g. *B.C.* Fig. 87. N. 2.) *subtending any of the Angles, as A is equal to the Squares of the other 2 sides* (*AB and AC*) *taken together, less 2 Rectangles* (*CAD*) *made by one side, containing the Acute-angle* (*CA*) *and its Segment AD reaching from the Acute-angle* (*A*) *to the Perpendicular* (*BE*). *let fall from the other side.*

Demonstration.

Make again $BC=a$, $AC=b$, $AB=c$, $AD=x$; then will $CD=b-x$. Therefore $cc-xx=\square BD$, and $aa-bb+2bbx-xx$ (*i. e.* $\square BC-\square CD$) will also be $=\square BD$.

Therefore $cc-xx=aa-bb+2bx-xx$.

i. e. (adding to both sides xx)

$cc=aa-bb+2bx$,

i. e. (adding on both sides bb, and subtracting $2bx$)

$cc+bb+2bx=aa$. Q. E. D.

CONSECTARYS.

I. IF in the last Equation, except one, you add on both sides bb, and subtract aa, you'l have $cc+bb-aa==2bx$, and, if moreover you divide hoth sides by $2b$, you'l have

$$\frac{cc+bb-aa}{2b}=x:$$ Which is the Rule, having 3 sides given in an Acute-angled Triangle, to find the Segment AD, and consequently the Perpendicnlar BD.

(*a*) *Eucl. Prop.* 13. *l.* 2.

Knowing therefore the Segments AD and CD, and also the Perpendicular BD in Oblique-angled Triangles, whether Obtuse-angled or Acute-angled, when moreover the sides BC and AB are likewise given, the Angles of either Right-angled Triangles or Oblique-angled ones, will be known; so that the last Case of Plane Trigonometry, which we deferr'd from *Prop.* 34 to this place, may hence receive its solution.

Proposition XLVII.

THE Square of the Tangent of a (α) *Circle, is equal to a Rectangle contain'd under the whole Secant DA, and that part of it which is without the Circle DE, whether the Secant pass thro' the Centre or not.*

Demonstration.

For in the first Case, if CB and CE are $= b$, DE $= x$, then will CD $= b + x$, & AD $= 2b + x$: therefore □ ADE $= 2bx + xx$ & □ CD $= bb + 2bx + xx$. therefore, if from the □ CD you substract □ CB $= bb$ the remainder will be $2bx + xx =$ □ BD $=$ □ ADE. Q. E. D.

(α) *Eucl.* 36. *Lib.* 3.

In the second Case, the lines remaining as before, make DE $= y$, FE or FA $= Z$: therefore the □ ADE will be $= 2zy + yy$, but the □ FC equal to the □ EC—□FE $= bb - zz$ *i. e.* $2bx + xx + zz$. But the same Square FD is $= zz + 2zy + yy$. Wherefore takingaway from these equal Squares the common one zz you'l have $2zy + yy = 2bx + xx$ *i. e.* pr. 1st Case $=$ □ BD. Q. E. D.

CONSECTARYS.

I. THerefore the Rectangles of diverse secants (as of ADE & *a d e* in *Fig.* 88, *n.* 1.) which are equal to the same Square. BD, are equal also to one another; which the last Equation in our Demonstration ($2zy + yy = 2bx + xx$) is an ocular proof of.

II. There-

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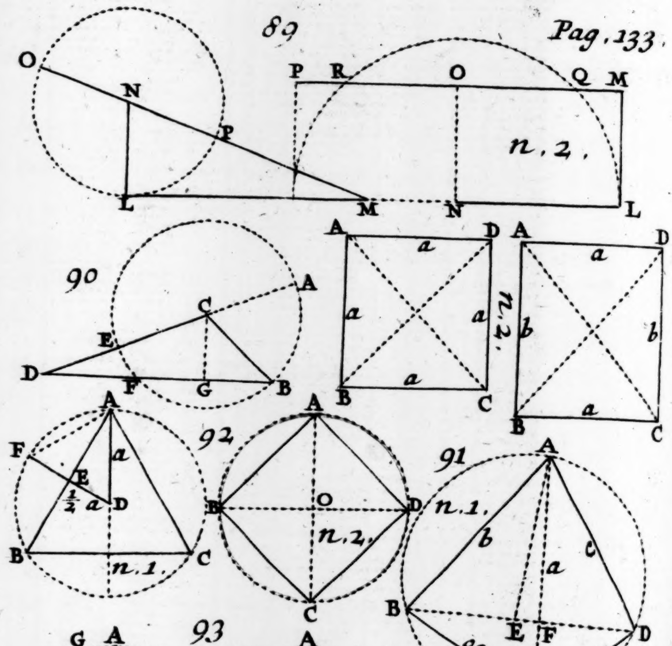

II. Therefore by *Prop.* 19. as *a* D to AD so is reciprocally DE to D *e* in the Fig. of the 2 Case.

III. Tangents to the same Circle from the same Point, as DB & D *b* (*n.* 3.) are equal; because the Square of each is equal to the same Rectangle.

IV. Nor can there be more Tangents drawn from the same point then two: For if besides DB & D *b*, D *c* could also touch the Circle, then it would be equal to them. By *Cons.* 2. but that is absurd by *Cons.* 3. *Def.* 7.

SCHOLIUM I.

HEnce is evident, the Original of the Geometrical Constructions, which *Cartes* makes use of, p. 6, and 7. in resolving these 3 Equations, $zz = az + bb$, $zz = az + bb$, & $zz = aZ = bb$. For in the First case, since he makes N O or N L (*Fig.* 89. 11. 1.) or N P $\frac{1}{2}$ *a*, and the Tangent z *M b*, make MNO through the Center *xx* will = Z the quantity sought; which thus appears: Making MO = Z, NM will = Z $\frac{1}{2}$ *a*, and its □ $ZZ - aZ + \frac{1}{4}aa$. But the □ OMP (which is by the present Prop. = □ LM, *i. e.* *bb*) together with the □ N L or NP *i. e.* $\frac{1}{4}aa$ is = □ NM by the *Pathagor. Theor.* Therefore $ZZ - aZ + \frac{1}{4}aa = bb + \frac{1}{4}aa$. *i e.* (taking away from both sides $\frac{1}{4}aa$) $ZZ - aZ = bb$ *i. e.* (by adding on both sides aZ) $ZZ = aZ + bb$: Which is the Equation proposed. In the Second Case, if you make PM = Z (as *Cartes* makes it) you'll have □NM + $ZZ + aZ + \frac{1}{4}aa$, and to this again as before you'l have $= bb + \frac{1}{4}aa$. Therefore $ZZ + aZ = bb$: Therefore $ZZ = -aZ + bb$: Which is the very Equation of the second Case. In the third Case, whether you make the w ole Secant RM (*N.* 2) or that part of it without the Circle QM = Z, the Root sought, there will come out on both sides the same Equation of the third Case, and so it is manifest, that this Equation has those 2 Roots. For if RM be = Z (adding + to the Fig of Cartes the Line NO which shall bisect QR, and makes OM = LN) OR or OQ will be $= Z - \frac{1}{2}a$, and so the □ $OQ = ZZ - aZ + \frac{1}{4}aa$, and this together with the ▭ RMQ (which is by vertue of the pres. Prop. = □ LM) = □NQ. *i. e.* OM, *i. e.* $ZZ -$

$aZ + \frac{1}{4}aa + bb = \frac{1}{4}aa$, *i. e.* (by adding aZ and taking away $\frac{1}{2}aa$) $ZZ + bb = aZ$; *i. e.* (taking away bb) $ZZ = aZ - bb$: Which is the the very Equation of the third Case. But if QM be made $= Z$, OQ or OR will $= \frac{1}{2} - Z$, and its $\square = \frac{1}{4}aa - aZ + ZZ$, as well as the former, and so all the rest. Q. E. D.

SCHOLIUM. II.

NOW if you would immediately deduce these Rules by the present Proposition, without the *Pythagorick Theorem*. It may (*e. g.* in the first Case) be done much shorter thus: If MO $= Z$ and NO or NP $= \frac{1}{2}a$, then will PM $= Za$: Therefore $\square$ OMP $= ZZ - aZ = bb$, or the $\square$ LM, by the pres. Proposition; by adding therefore to both sides aZ, you'l have $ZZ = aZ + bb$, which is the very Equation of the first Case. In the second Case, if MR be Z, QM or PR will $= a - Z$: Therefore $\square$ RMP $aZ - ZZ = bb$, *i. e.* $aZ = bb + ZZ$, *i. e.* $aZ - bb = ZZ$; but if QM $= Z$, RM will $= a - Z$. Therefore $\square$ RMP $aZ - ZZ = bb$, as before, &c.

SCHOLIUM III.

FRom the 2 Confect. of the present Proposition, flows another Rule for solving the last Case of *Plain Trigonometry*, which we solved in the Confect. of the foregoing Prop. *viz.* If you have all the three sides of the Oblique-angled Triangle BCD (*Fig.* 90) given, if from the Center C, at the distance of the lesser side CB you describe a Circle, then will, by Confect. 2. of the present Proposition, BD the Base of the Triangle (here we call the greatest side of the Triangle the Base, or in an Equicrural Triangle, one of the greatest) will be to AD, (the sum of the Sides DC + CB) as DE the difference of the Sides, to DF the Segment of the Base without the Circle; which being found, if the remainder of the Base within the Circle be divided into two equal parts, you'l have both FG and GB, as also DG; which being given, by help of the Right-angled $\triangle\triangle$ GBC and GDC all the Angles required may be found.

Propo-

Proposition XLVIII.

IN *any Quadrilateral Figure* (a) *ABCD* (Fig. 91. N. 1.) *inscribed in a Circle, the* ▭ *of the Diagonals AC and BD is equal to the two Rectangles of the opposite sides AB into CD, and AD into BC.*

Demonstration.

Having drawn AE so that the Angle BAE shall be equal to the Angle CAD, the Triangles thereby formed (for they have the other Angles EBA and ACD in the same Segment equal, by vertue of Consect. 1. Prop. 33.) will be Equiangular one to another and consequently (by Prop. 34.) as AC to CD so AB to BE. Wherefore by making AC = *a* and CD = *e a*, and AB = *b*, BE will be = *e b*. In like manner when in the △ △ BAC and EAD, the respective Angles are equal (*viz.* adding the common part EAF to BAE and CAD, equal by Constr.) and besides the angles BCA and EDA in the same Segment are also equal; these Triangles will also be Equiangular, and AD will be to DE as AC to CB; wherefore by putting, as before, *a* for AC, and *oa* for CB, and *c* for AD, DE will = *o c*. Therefore the whole BD = *e b* + *o c*. The Rectangle therefore of AC into BD will = *e b a* + *o c a* = ; the Rectangle of AB into CD = *e b a* + ▭ of AD into BC = *o a c*. Q. E. D.

(a) Ptol. lib. 1. Almagest.

SCHOLIUM

IN Squares and Rectangles (N. 2.) the thing is self-evident: For in Squares if the side be *a*, the Diagonals AC and BD will be $\sqrt{2aa}$, and so their Rectangle = $2aa$ will be manifestly equal to the two Rectangles of the opposite sides. In Oblongs, if the two opposite sides are *a* and the others *b*, the Diagonals will be $\sqrt{aa+bb}$, and their Rectangle $aa+bb$ manifestly equal to the two Rectangles of the opposite Sides.

Proposition XLIX.

THE *side* (AB) *of an Equilateral Triangle* (ABC, Fig. 92. N. 1.) *inscribed in a* (a) *Circle, is in Power triple of the Radius* (AD) i. e. of the □ *of AD.*

Demonstration.

MAke *AD* or *FD* $= a$, and so its Square *aa*. Since Therefore, having drawn DF thro' the middle of *AB*, or the middle of the Arch AFB let DE be $= \frac{1}{2} a$; for the angles at E are right ones, by *Consect.* 5. *Definit.* 8. and the Hypothenuses AD, AF, are equal, by Schol. of Definition 15. but the side AE is common. Therrefore the other Sides FE and ED are equal by, by *Consect.* 3. *Prop.* 43.) and the □ of the latter is $\frac{1}{4} aa$, which subtracted from *aa* leaves $\frac{3}{4} aa$ for the □ of AE. Therefore the line AE is $\sqrt{\frac{3}{4} aa}$, and consequently AB $2\sqrt{\frac{3}{4} aa}$, *i. e.* $\sqrt{\frac{12}{4} aa}$, *i. e.* $\sqrt{3aa}$: therefore □ AB $= 3aa$. Q. E. D.

CONSECTARYS.

I. IF the Radius of a Circle be $= a$, the side of an Inscribed Regular Triangle will be $\sqrt{3aa}$, *e. g.* if AD be 10, A*B* will be $\sqrt{300}$; and if AD be 10, 000, 000, A*B* will be $\sqrt{300, 000, 000, 000, 000}$, *i. e.* 17320508, and the Perpendicular DE 5000, 000.

II. Hence it is evident, that in the genesis of a *Tetraedrum* proposed in *Def.* 22, that the elevation CE (*Fig.* 44, *N.* 1.) is to the remaining part of the Diameter of the Sphere CF as 2 to 1; for making the Radius C*B* $= a$ and its □ *aa*, the □ of AB or *B*E will $= 3aa$, by the present Proposition. Therefore the □ of C*B* being subtracted from the □ of *B*D or *B*E, there remains the □ of CE $= 2aa$. *But* since CE, C*B*, CF, are continual Proportionals, by *N.* 3. *Schol.* 2. *Prop.* 34. CE will be to CF as the □ of CE to the Square of C*B*, by vertue of Prop. 35. *i. e.* as 2 to 1.

(*a*) *Eucl.* 13 *lib.* 12.

SCHOLIUM.

HEnce you have the *Euclidean* way of generating (*a*) a *Tetraedrum*, and inscribing it in a given Sphere, when he bids you divide the Diameter EF of a given Sphere so that EC shall be 2 and CF 1, and then at EF to erect the Perpendicular CA, and by means thereof to describe the Circle A*B*D, and to inscribe therein an Equilateral Triangle, *&c.*

Proposition

Proposition L.

THE *Side* (*AB*) *of a Regular Tetragon* (*or Square*) (*ABCD*, Fig. 92. N. 2.) *is double in Power of the Radius* (*AD*).

Demonstration.

For having drawn the Diameter AC and *B*D, the Triangle AO*B* is Right-angled, and consequently, by the *Pythagorick Theorem*, if the □ of AO and *B*O be made equal to *aa*, then will the □ of A*B*$=2aa$, Q. E. D.

CONSECTARY.

THerefore when the Radius of the Circle AO is made $=a$, the side of the □ A*B* will $=\sqrt{2aa}$, *e.g.* if AO be 10, A*B* will be $\sqrt{200}$; and if AO be 10, 000, 000, A*B* will be $\sqrt{200,000,000,000,000}$, *i. e.* 14142136.

Proposition LI.

THE *side AB of a Regular Pentagon* (α) (*ABCDE*) (Fig. 93. N. 1.) *is equal in Power to the side of an Hexagon and Decagon inscribed in the same Circle*, i. e. *the* □ *of AB is equal to the Squares AF and AO taken together.*

Demonstration.

Make A O$=a$ and A F$=b$, A*B*$=x$:
(α) Eucl. Prop. 13. lib. 13. We are to demonstrate that $xx=aa+bb$: which may be done by finding the side A*B* by
(α) Eucl. Prop. 10. lib. 13. the parts *B*H and HA, after the following way: First of all the angle AO*B* is 72°, and the others in that Triangle at A and *B* 54°. But *B*GG is also 54°, as subtending the Decagonal Arch *B*F of 36°, and also one half of it FG of 18°. Therefore the △△ A*B*O and H*B*O are Equiangular, and you'l have

A3

As AB to BO so BO to BH

$$x - a - a - \frac{aa}{x}$$

Secondly, in the Triangle BFA the Angles at B and A are equal by *N.* 3. *Consect*. 5. *Def.* 8. and by vertue of the same also the Angles at F and A in the $\triangle$FHA are so too. Wherefore the $\triangle\triangle$ BFA and FFA are Equiangular, and you'l have

As BA to AF so AF to AH

$$x \text{ to } b - b - \frac{bb}{x}$$

Therefore the whole line AB (because the part AH is found $= \frac{bb}{x}$ and BH$= \frac{aa}{x}$) will be $\frac{aa+bb}{x}$, which was first made $=x$; so that now $\frac{aa+bb}{x}$ is $=x$, and multiplying both sides by x, $aa+bb=xx$. Q.E.D.

CONSECTARY I.

THerefore if the Radius of a Circle be (a) the side of a Pentagon AB will be $\sqrt{aa \times bb}$.

CONSECTARY II.

THerefore the $\square$ AI$=\frac{aa+bb}{4}$ and $\square$ OI$=\square$OA$-\square$AI $= aa - \frac{aa-bb}{4}$ i.e. $\frac{3aa-bb}{4}$. Therefore OI$=\sqrt{\frac{3aa-bb}{4}}$. Which yet may be expressed otherwise, *viz.* OI $=\frac{a+b}{2}$.

Demonstration.

(a) *Eucl. Prop.* 1. *lib.* 14.

Make (a) OA or OF ($N.$ 2.) as before $=a$, AF$=b$, and FI now $=x$; then will OI$=a-x$ and having drawn the Arch FK at the Interval AF, so that AK may be equal to this, and FI$=$ IK

IK $=x$; then will the angle IKA $=$ F72°. Therefore the angle AKO $=$ 108; and since KOA is 36°, KAO will be also 36°, and so KA $=$ KO $=$ AF $=b$. Wherefore OI is $=b+x$, which was above $a-x$. Therefore 2OI $=a-x+b+x$, i. e. $a+b$. Therefore OI $=\frac{a+b}{2}$.

CONSECTARY III.

THerefore the difference between the Perpendicular of the Triangle DE (*Fig.* 92 and 93. *N.* 1.) and the Perpendicular of the Pentagon OI is $=\frac{1}{2}b$, by vertue of *Consect.* 2. of this and of the Demonstrat. of *Prop.* 49.

CONSECTARY IV.

HEnce is also evident, by the present Proposition, that which in *Fabri's* Genesis of an *Icosaedr. Def.* 22. we said, *viz.* that (See *Fig.* 46.) B*a* is equal to the side of a Pentagon *B*A, because, *viz.* *Fa* is $=$ to the Semidiameter OB, and BF is the side of a Decagon.

Proposition LII.

T*HE side of an Hexagon is in Power equal to the Radius, as being it self equal to it by* N. I. Schol. Def. 15.

Proposition LIII.

T*HE side of a Regular Octagon* (*ABCD*, &c. *Fig.* 94.) *is equal in Power to half the side of the Square, and the difference* (*PB*) *of that half side from the Radius, taken together.*

Demonstration.

For that the □ of AB is $=$ to the □ of AP $+$ □ BP, is evident from the *Pythag. Theor.* But that PO is $=$ PA half the side of the Square, is evident from the equality of the Angles PAO and POA, since each is a half right one or 45°. Wherefore

fore the side of the Octagon is equal in Power to half the side of the Square, &c. Q.E.D.

CONSECTARY.

THerefore, if the Radius be $= a$, AP will be, by vertue of the *Pythag. Theor.* $= \sqrt{\frac{1}{2}aa}$ and P$B = a - \sqrt{\frac{1}{2}aa}$ of that $\square \frac{1}{2}aa$; and of this $\square 1\frac{1}{2}aa - \sqrt{\quad 2}$. Therefore the Sum of $\square$ AB $= 2aa - \sqrt{2a^4}$. Therefore the side of the Octagon $= \sqrt{2aa - \sqrt{2a^4}}$. *e. g.* If AO be 10, AB will be $\sqrt{200 - \sqrt{20000}}$, and if the Radius AO 10000000, AB will be $\sqrt{200,000,000,000,000 - \sqrt{200000000000000000000000000}}$, by vertue of the present Conf. or from the Prop. it self, if AO be 10000000, AP will be vertue of the Conf. of Prop. 50. $= 7071068$, and consequently BP $= 2928932$. The Squares of these added together give the $\square$ AB, and the Root thence extracted A$B = 7653668$.

Proposition LIV.

THE side of a Regular Decagon (a) *is equal in Power to the greatest part of the side of an Hexagon cut in mean and extreme Reason.*

Demonstration.

(a) *Eucl.* 9. *lib.* 13. *Corol.*

Suppose BD (*Fig.* 95.) divided in mean and extreme Reason in E, and BA to be joined to it long ways = to the side of a Decagon inscribed in the same Circle, whose Radius is BC or AC = BD. Now we are to demonstrate that DE the greatest part of the de of the Hexagon BD divided in mean and extreme Reason, is equal to the side of a Decagon BA, and the Power of the one equal to the Power of the other. Because the Angle ACB is 36°, ABC and A 72°, and consequently CBD 108°, BCD and D will be each 36°, and so the whole ACD 72°. (that so CD may pass precisely thro' the other end of the side of the Pentagon

tagon AF.) Wherefore the △△ ABC and ADC are Equiangular, and

AD to AC i. e. BD } as AC i.e. BD } to AB. Therefore the whole line AD is divided in mean and extreme Reason. But BD is also divided in the same Reason by Hyp. Wherefore

As AD to DB and DB to BA,
So DB to DE and DE to EB.

Therefore DB is in the same Proportion to DE as DB to BA. Therefore DE is $=$ BA, and the Power of the one to the Power of the other. Q. E. D.

CONSECTARYS.

THerefore, if the Radius of the side of the Hexagon is a the side of the Decagon will be $\sqrt{\frac{5}{4}aa} - \frac{1}{2}a$, by *Schol*. 2. *Prop*. 27. *e. g.* if the Radius be 10, the side of the Decagon will be $\sqrt{125} - 5$, and if the Radius be put 10 000 000, the side of the Decagon will be $= \sqrt{125\ 000\ 000\ 000\ 000} - 5000\ 000$, *viz*. by adding the Square of the Radius and the Square of half the Radius into one Sum; whence you'l have the side of the Decagon $= 6180340$; the half whereof 3090170 gives the difference between the Perpendiculars of the Triangle and the Pentagon, by *Conf*. 3. *Prop*. 51.

II. The side therefore of the Pentagon is by *Prop*. 51. $\sqrt{\frac{5}{2}aa - \sqrt{\frac{5}{4}a^4}}$; for the Square of the Hexagon is aa or $\frac{2}{2}aa$, the □ of the Decagon $\frac{3}{2}aa - \sqrt{\frac{5}{4}a^4}$: the Root extracted out of the Sum of these is the side of the Pentagon, *viz*. $\sqrt{\frac{5}{2}aa - \sqrt{\frac{5}{4}a^4}}$ *e. g.* if the Radius be 10, the side of the Pentagon will be $\sqrt{250 - \sqrt{125000}}$, and if the Radius be put 10 000 000, since the side of the Hexagon is equal to it, and the side of the Decagon 6180340, their Squares being added into one Sum, the Root extracted out of that Sum will give the side of the Pentagon, 1755704 nearly; and the sides being collected into

one

one ſum, the half of it 8090170 will give the Perpendicular in the Pentagon OI, by *Conſect.* 2. *Prop.* 51.

SCHOLIUM I.

TO illuſtrate what we have deduced in the Conſectarys of Prop. 51, you may take the following Notes. If a be put $= 10$ or $\frac{1000}{100}$, the ſide of the Decagon will be $= \sqrt{125} - 5$, *i.e.* $\frac{618}{100}$ nearly $= b$; therefore the $\square\, aa = \frac{1000000}{10000}$ and the $\square\, bb = \frac{381927}{10000}$: therefore $aa + bb$ or the $\square$ AB $= \frac{1381924}{10000}$, the Perpendicular OI $= \frac{a+b}{2} = \frac{1618}{100}$ divided by 2, that is, $\frac{809}{100}$. Now the $\square$ of AI is $\frac{1}{4}$ of the $\square$ AB $= \frac{355481}{10000}$ the $\square$ OI $= \frac{aa + 2ab + bb}{4} = \frac{654481}{10000}$. Now if you add the $\square$ AI and the $\square$ OI, the ſum will be $= \square$ AO $= \frac{2aa + 2ab + 2bb}{4}$, *i. e.* $\frac{aa + 2ab + bb}{2} = \frac{99}{1} | \frac{996}{0000}$, or near 100. Thus likewiſe, ſince the Perpendicular above found OI, in *Conſect.* 1. may be alſo determined by $\frac{\sqrt{3aa - bb}}{4}$, ſince aa is $= \frac{1000000}{10000}$, and $3aa$ $3\,\frac{0000000}{10000}$, ſubtracting from it $bb = \frac{381924}{10000}$ the Remainder will be $\frac{2618076}{10000}$, and this being divided by 4, you'l have the $\square$ OI $= \frac{654519}{10000}$. and the Root of it extracted $\frac{809}{100}$ nearly; ſo that thoſe two different quantities in Conſect. 1. will rightly expreſs the ſame Perpendicular OI.

SCHOLIUM II.

NOW therefore as we have Practical Rules to determine Arithmetically the ſides of the Pentagon and Decagon, ſo alſo they may be found Geometrically by what we have demonſtrated. For if the Semidiameter C*B* (*Fig.* 96. *N.* 1.) be divided into 2 parts, EC will $= \frac{1}{2}a$; and erecting perpendicularly the Radius CD $= a$ DE will $= \sqrt{\frac{5}{4}aa}$. Moreover if you cut off EF equal to it, FC will be $= \sqrt{\frac{5}{4}aa} - \frac{1}{2}a =$ to the ſide of the Decagon, by *Conſect.* 1. Having therefore drawn DF, which is equal in Power to the Radius or Side of the Hexagon DC, and the ſide of the Decagon FC together, by the *Pythag. Theorem* it

it will be the side of the Pentagon sought. Much to the same purpose is also this other new Construction of the same Problem, wherein BG (*Numb.* 2.) is the side of the Hexagon BD the side of the Square, to which GF is made equal, so that FC is that side of the Decagon, and DF of the Pentagon; which we thus demonstrate after our way: Having bisected GH the side of an Equil. Triangle, the Square of GE will be $\frac{3aa}{4}$, by *Prop.* 48. which being subtracted from the Square of GF $=$ $2aa$, *viz.* $\frac{8}{4}aa$, by *Prop.* 49. there will remain for the Square of EF $\frac{5}{4}aa$, and for the line EF $\sqrt{\frac{5}{4}aa}$, and for FC $\sqrt{\frac{5}{4}aa} - \frac{1}{2}a$, which is the side of the Decagon, as DF of the Pentagon, after the same way as before.

Proposition LV.

The side of a Quindecagon (or 15 sided Figure) is equal in Power to the half Difference between the side of the Equil. Triangle and the side of the Pentagon, & moreover to the Difference of the Perpendiculars let fall on both sides taken together.

Demonstration.

For if AB (*Fig.* 97.) be the side of an Inscribed Triangle, and De the side of a Pentagon parallel to it; AD will be the side of the Quindecagon to be inscribed, by *Consect.* 4. *Def.* 15. But this side AD in the little Right-angled Triangle, is equal in Power to the side AH (which is the half Difference between AB and DE) and the side HD (which is the difference between the Perpendiculars CF and CG) taken together by the *Pythag. Theor.* Q. E. D.

CON-

CONSECTARY.

HEnce if we call the Side *AB* of the Equil. △ *c*, and make the side of the Pentagon DE $= d$, AH will $= \frac{c - d}{2}$ while HD is $= \frac{1}{2} b$, by *Consect.* 3. of *Prop.* 51. Since therefore the △ AH is $= \frac{cc - 2\,c \times d\,d}{4}$ and the △ H D $= \frac{bb}{4}$ the △ AD will $\frac{cc - 2\,c\,d \times b\,b}{4}$

Therefore the side of the *Quindecagon* $= \sqrt{\frac{cc - 2\,c\,d \times d\,d \times bb}{4}}$ that is, Collecting the Square of the half difference of the sides of the Triangle and Pentagon, and the Square of the difference of the Perpendiculars into one Sum, and then Extracting the Square Root of that Sum, you'l have the side of the Quindecagon sought. *E.* 9. if rhe Radius CI be made 10000000 the difference of the sides of the Triangle 17320508, and of the side of the Pentagon 11757704 will be 5564804, and the half of this 2782402; but the difference of the Perpendicular CF from the Perpendicular CG, is 3090170.

The Squares therefore of these Two last Numbers being Collected into one Sum, nnd the Root Extracted will give the side of the Quindecagon 4158234 nearly.

SCHOLIUM.

HEre we will shew the Excellent use of these last Propositions in making the Tables of Signs. For having found above, supposing the Radius of 10000000 parts, the sides of the chief Regular Figures, if they are Bisected, you will have so many Primary Sines; *viz.* from the side of the Triangle the side of 60 Degrees 8660754, from the Side of the Square, the Sine of 45° 7071068; from the Side of the Pentagon, the Sine of 36° 5877853; from the Side of the Hexagon, the Sine of 30° 5000000; from the Side of the Octagon, the Sine of 22° 30 3826843; from the Side of

of the Decagon the ſine of $18^\circ = 3090170$; from the ſide of the quindecagon laſtly the ſine of $12^\circ = 2079117$. From theſe ſeven primary ſines you may find afterwards the reſt, and conſequently all the Tangents and Secants according to the Rule we have deduc'd, *n.* 3. *Schol.* 5. *Prop.* 34. and which *Ph. Lansbergius* illuſtrates in a prolix Example in his Geom. of Triangles *Lib.* 2. *p.* 7. and the following. But after what way, having found theſe greater numbers of ſines, Tangents, *&c.* Logarithms have been of late accommodated to them, remains now to be ſhewn, which in brief is thus; *viz.* the Logarithms of ſines, *&c.* might immediately be had from the Logarithms of vulgar numbers, if the tables of vulgar numbers were extended ſo far, as to contain ſuch large numbers; and thus the ſine *e. g.* of 0 gr. 34. which is 98900 the Logarithm in the Chiliads of *Vlacquus* is 4995196 2916. But becauſe the other ſines which are greater than this are not to be found among vulgar numbers (for they aſcend not beyond 100000, others only reaching to 10000 or 20000) there is a way found of finding the Logarithms of greater numbers, than what are contained in the Tables. *E g.* If the Logarithm of the ſine of 45° which is 7071068 is to be found, now this whole number is not to be found in any vulgar Tables, yet its firſt four notes 7071 are to be found in the vulgar Tables of *Strauchius* with the correſpondent Logarithm 3. 8494808, and the five firſt 70710 in the Tables of *Vlacquus* with the Log. 4. 8494808372. One of theſe Logarithms, *e g.* the latter, is taken out, only by augmenting the Characteriſtick with ſo many units, as there remain notes out of the number propoſed, which are not found in the Tables, ſo that the Log. taken thus out will be 6. 8494808372. Then multiply the remaining notes of the propoſed number by the difference of this Logarithm from the next following, (which for that purpoſe is every where added in the *Vlacquian* Chiliads, and is in this caſe 61419) and from the Product 4176492 caſt away as many notes as adhere to the propoſed number beyond the tabular ones, in this caſe 2; for of the remainder 41764, if they are added to the Logarithm before taken out, there will come the Logarithm required 6. 8494850136, *viz.* according to the Tables of *Vlacquus*, wherein for the Log. of 10 you have 10, 000, 000,

L 000;

000; but according to those of *Strauchius* which have for the Logarithm of 10 only 10, 000, 000, you must cut off the three last notes, that the Logarithm of the given sine may be 6. 8494850; as is found in the *Strauchian* and other tables of sines, except that instead of the Characteristick 6 there precedes the Characteristick 9, whereof we will add this reason: If the Characteristicks had been kept, as they were found by the rule just now given, the Logarithm of the whole sine (which is in the *Strauchian* Tables 10, 000, 000) would have come out 70, 000, 000, incongruous enough in Trigonometrical Operations. Wherefore that Log. of the whole sine might begin from 1, for the easiness of Multiplication and Division they have assumed 100, 000, 000; the Characteristick being augmented by three, wherewith it was consequently necessary to augment also all the antecedent ones; and hence *e. g.* the Logarithm of the least sine 2909 begins from the Characteristick 6, which otherwise according to the Tables of vulgar numbers would have been 3.

Having found after this way the Logarithms of all the sines (altho' here also if you have found the Logarithms of the signs of 45° and moreover the Logarithm of 30, the Logarithms of all the rest may be compendiously found by addition and substraction from a new principle which now we shall omit) the Logarithms of the Tangents and Secants may easily be found also, only by working, but now Logarithmically, according to the Rules of *Schol.* 5. *Prop.* 34. *n.* 5. and 6.

Proposition LVI.

(a) Eucl. 13. lib. 13.

THE side of a Tetraëdrum (a) *or equil. Pyramid is in power to the Diameter of a circumscribed Sphere, as* 2 *to* 3.

Demonstration.

For because by the genesis of the *Tetraëdrum Def.* 22. (see its *Fig.* 44 *n.* 1.) and *Schol. Prop.* 49. OC is $\frac{1}{3}$ of the semidiameter OB, which we will call a the □ of CB will be $= \frac{8}{9}aa$ by the *Pythag. Theor.* and so the power (or Sq.) of the side of the *Tetraëdrum* $= \frac{24}{9}aa$ by *Prop.* 49. but the power

er of the Diam. $2a$ or $\frac{6}{3}a$ is $\frac{36}{9}aa$. Therefore the power of the ſide of the *Tetraëdrum* is to the power of the Diam. as 24 to 36. *i. e.* (dividing each ſide by 12) as 2 to 3. Q. E. D.

Or more ſhort.

The □ of CB is = 2 by *Schol* of *Prop.* 49 and the □ of EC = 4. Therefore the □ EB = 6. But the □ EF is = 9. Therefore the □ of EB is to the □ of EF as 6 to 9, *i. e.* as 2 to 3. Q. E. D.

CONSECTARY.

THerefore if the Diam. EF be made = a, the ſide EB will be = $\sqrt{\frac{2}{3}aa}$.

Propoſition LVII.

THE ſide of the Octaëdrum (α) is in power one half of the Diameter of the circumſcribed Sphere.

(α) *Eucl.* 14. *lib.* 13.

Demonſtration.

For ſince by the geneſis of the *Octaëdrum Def* 22. (ſee *Fig.* 44. *n.* 2.) CA, CB, CF, *&c.* are ſo many radii of great circles, if for Radius you put a, the ſquare of AF will be = $2aa$ by the *Pythag. Theor.* But the ſquare of the Diam. FG = $2a$ is $4aa$. Therefore the power of the ſide is to the power of the Diam as 2 to 4, *i. e.* as 1 to 2. Q. E. D.

More ſhort.

Becauſe AF is alſo the ſide of a ſquare inſcribed in the greateſt circle by the gen. of the *Octaëdrum*; the □ of AF will be by *Prop.* 50. to the □ of FC as 2 to 1: Therefore to the ſquare of FG as 2 to 4, by *Prop.* 35. Q. E. D.

CONSECTARY.

THerefore if the Diam. of the sphere be made (a) the side of the *Octaëdrum* AF will be $\sqrt{\frac{1}{2}aa}$.

Proposition LVIII.

(α) Eucl. 15. lib. 13.

THE side of the Hexaëdrum or Cube (α) is in Power subtriple of the diameter of the circumscribed sphere.

Demonstration.

Making a the side of the inscribed cube GF or FE (*Fig.* 98.) the square of the diagonal GE of the base of the cube will be $2aa$ by the *Pythag. Theor.* and by the same Reason the square of the diam of the cube and the circumscribed sphere GD will be = to the square of GE + □ DE = $3aa$. Q. E. D.

CONSECTARYS.

I. THerefore if the diameter of the sphere be made = a, the side of the cube AB will be $\sqrt{\frac{1}{3}aa}$.

II. The diameter of the sphere is equal in power to the side of the *Tetraëdrum* and cube taken together. For if the power of the diam. of the sphere be made aa the power of the side of the *Tetraëdrum* will be $\frac{2}{3}aa$ by *Consect. Prop* 56. and the power of the side of the cube $\frac{1}{3}aa$ by the pres. *Consect.* 1. Wherefore these two powers jointly make aa. Q. E. D.

Proposition LIX.

(α) Eucl. Consect. 1. Prop. 17. lib. 13.

THE side of the Dodecaëdrum (α) is equal in power to the greater part of the side of the cube divided in mean and extreme reason.

Demon-

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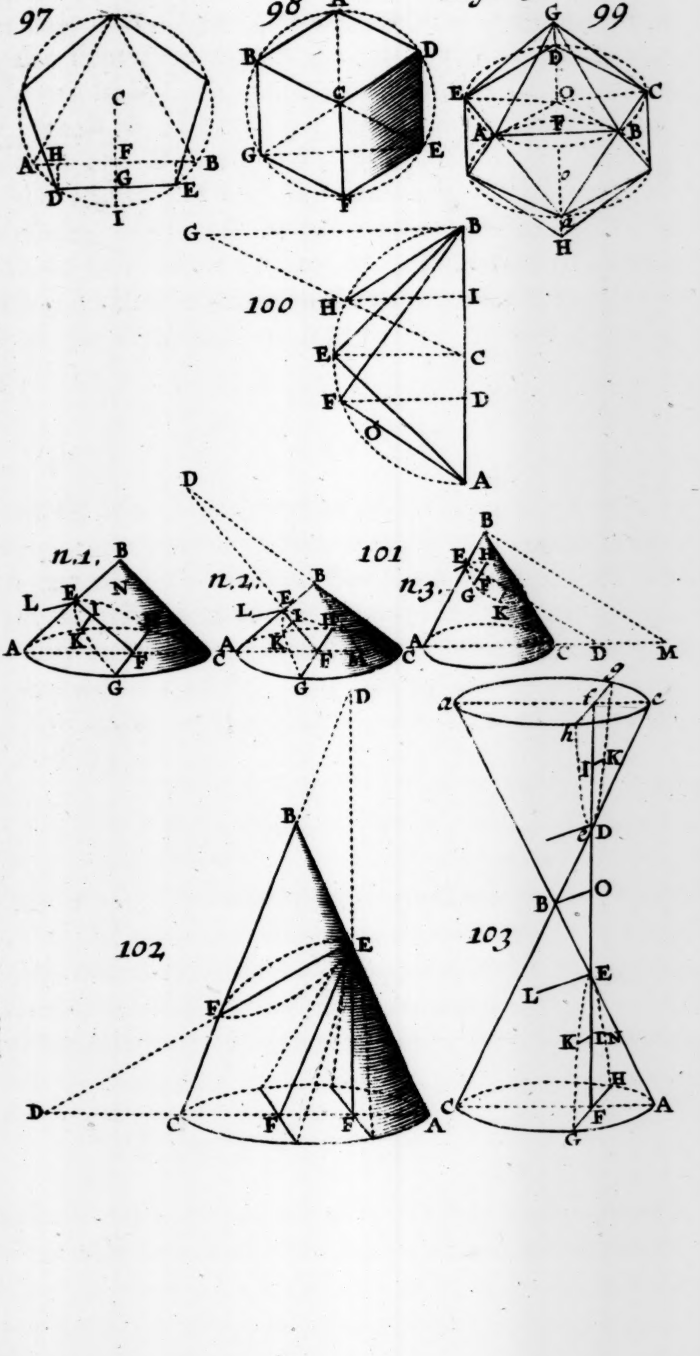

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Demonstration.

For if to the side of the cube AB (*vid. Fig.* 45. *n.* 1.) and to its upper base ABCD you conceive to be accommodated or fitted a regular Pentagon according to the genesis of a *Dodecaëdrum* laid down in *Def.* 22, and at the interval B*e* you make the arch *ef*, the △ △ AB*e* and A*ef* will be equiangular; (for the angles at A and B being 36°, and A*e*B 108, having drawn *ef*, the angles B*ef* and B*fe* are each 72°; therefore A*ef* the remaining angle will be 36°) wherefore as AB to B*e* (*i. e.* B*f*) so A*e* (*i. e.* B*e* or B*f*) to A*f*. Therefore the side of the cube AB is divided in mean and extream reason in *f*, and B*e* the side of the *Dodecaëdrum* is ⊏⊐ to the greater part B*f*. Q. E. D.

SCHOLIUM.

HEnce would arise a new method of dividing a given line in mean and extream reason, *viz.* if you apply to the given line a part of the equilateral Pentagon by means of the angles A and B 36°, and at the interval B*e* you cut off B*f*. This angle may be had geometrically, if another regular Pentagon be inscribed in a circle, and having drawn also a like subtense, if the angles at the subtense are made at A and B equal, by *Eucl.* 23. *lib.* 1.

Proposition LX.

THE side of an Icosaëdrum (a) is equal in power to the side of a Pentagon in a circle containing only five sides of the Icosaëdrum; and the semi diameter of this circle is in power subquintuple of the Diam. of the sphere of the circumscribed Icosaëdrum.

(*a*) *Eucl. Prop.* 16. *Coroll. lib.* 13.

Demonstration.

Both these are evident from the genesis of the *Icosaëdrum* in *Def.* 22. The first immediately hence, because all the other sides

ſides of the triangles (*Fig.* 99.) A*a*, B*a*, *&c.* are made equal to the ſide of the Pentagon AB by *Conſect.* 4. *Prop* 51. The latter from this inference; if for OA the radius of the circle you put *a* (ſince the ſide of the Pentagon, which here is alſo the ſide of the *Icoſaëdrum*, it will be equal in power to the radius and ſide of the Decagon taken together by the aforeſaid *Prop.*) the altitude OG will be the ſide of the Decagon $= \sqrt{\frac{5}{4}aa} - \frac{1}{2}a$ by *Conſect.* 1. *Prop.* 54. to which the equal inferior part *o*H being added, and the intermediate altitude O*o* $= a$, you'l have the whole diameter of the circumſcribed ſphere GH $= a + 2\sqrt{\frac{5}{4}aa} - \frac{2}{2}a$ i. e. $2\sqrt{\frac{5}{4}aa}$ i. e. $\sqrt{\frac{20}{4}aa}$ i. e. $\sqrt{5aa}$ and ſo the ſquare of the diameter of the ſphere will be $5aa$: Therefore the ſquare of the diameter of the ſphere is to the ſquare of the ſemi-diam. of the circle containing the five ſides of the *Icoſaëdrum* as 5 to 1. Q. E. D.

SCHOLIUM.

IT is alſo evident that a ſphere deſcribed on the diameter GH will paſs thro' the other angles of this *Icoſaëdrum*; for aſſuming the center between O and *o* the radius FG will be $= \sqrt{\frac{5}{4}aa}$. But FA is alſo $= \sqrt{\frac{5}{4}aa}$; for the □ of FO is $= \frac{1}{4}aa$, and the □ AO $= aa$: Therefore the ſum is $= \frac{5}{4}aa$ $=$ □ FA. Q. E. D.

CONSECTARY I.

THerefore, if the radius of the circle ABCDE remain *a*, you'l have the altitude OG $\sqrt{\frac{5}{4}aa} - \frac{1}{2}a$, and the ſide of the *Icoſaëdrum* $\sqrt{\frac{5}{2}aa - \sqrt{\frac{5}{4}a^4}}$, by *Conſ* 1. and 2. *Prop.* 54. and the diam of the circumſcribed Sphere $2\sqrt{\frac{5}{4}aa}$, as is evident from the Demonſtration.

CON-

CONSECTARY II.

Being a general one of the five last Propositions.

(*a*) *Eucl. Prop.* 18. *and last, lib.* 13.

IF AB (*Fig.* 100) be the diameter of a ſphere (*a*) divided in D ſo that AD ſhall be $\frac{1}{3}$ AB, then (having erected the perpendicular DF) BF will be the ſide of the *Tetraëdrum* by *Prop.* 56. and AF the ſide of the *Hexaëdrum* by *Prop.* 58. *Conſ.* 2. and BE or AE (erecting from the center the perpendicular CE) will be the ſide of the *Octaëdrum* by *Prop.* 57. Now if AF be cut in mean and extreme reaſon in O, you'l have AO the ſide of the *Dodecaëdrum* by *Prop.* 59. Laſtly, if you erect BG double of CB, HI will be double of CI, and the □ of HI = 4 □ of CI; conſequently the □ CH or CB = 5 □ CI. Therefore the □ of AB (double of CH) is alſo = to 5 □ of HI (which is double of CI) therefore HI is the radius of the circle circumſcribing the Pentagon of the *Icoſaëdrum*, and IB the ſide of the *Decagon* inſcribed in the ſame circle, and HB the ſide of the *Pentagon*, and alſo the ſide of the *Icoſaëdrum*, by *Prop.* 60.

The End of the first Book.

THE

THE SECOND BOOK.

SECTION I.

Containing

DEFINITIONS.

Definition I.

IF a Cone ABC (*Fig.* 101.) be conceived to be cut by a plane at right angles to the side of the cone, *e. g.* BA; the Plane EFGHE arising by this section, and terminated on the external surface of the cone by the curve line HEG, *&c.* was anciently by *Euclid*, *Archimedes*, *&c.* called the *Conick Section*; and if the sides of the cone AB and BC made a right angle at B, as *n.* 1. the section was particularly called the *Section* of a *right-angled Cone*; but if it made an obtuse angle, as *n.* 2. it was called the *Section of an obtuse-angled Cone*; if, lastly, it made an acute one, as *num.* 3. it was called (3.) the *Section of an acute-angled Cone.*

Definition II.

BUT afterwards their Successors, particularly *Apollonius Pergæus*, found from the properties of these Curves, which their Predecessors had happily discovered, that the same (all of them) might be generated in one and the same cone whether right-angled, obtuse-angled, or acute-angled, if the section EF (*Fig.* 102.) is made in the first case parallel to the

the oppoſite ſide BC; in the ſecond caſe, if it meet that ſide produced upwards; for the third, when it meets downwards with the diameter of the baſe AC produced to D. And to give new names (for the old ones would do no more now) to theſe Sections, to diſtinguiſh them one from another, nominating them from their Properties hereafter demonſtrated, they called the firſt a *Parabola*, the ſecond an *Hyperbola*, the third an *Ellipſis*.

Definition III.

BUT it is evident, that only the plane making the ſection of the ſecond caſe, being according to the line FED produced or carried on, (*Fig.* 103.) will fall upon the vertical Cone *a*B*c* comprehended under the ſides AB, CB, *&c.* continued onwards, and there produce another Section oppoſite to the former; whence theſe, *viz.* GEHG and *gehg* are called *oppoſite Sections.*

Definition IV.

BEſides theſe names of the ſections, there are ſeveral others made uſe of to denote various lines drawn and conſidered both within and without thoſe ſections, the chief whereof we ſhall here explain. And firſt of all, in general the line EF ſo let fall thro' the middle of the ſection from its top E (which is called the *Vertex* of the ſection) to the diameter of the baſe of the Cone AC (produced if occaſion be) that it ſhall biſect the line GH and all others parallel to it, is called the *Diameter* of that *Section*; and particularly it is called the *Axis* of the *Section* if it cuts them at right angles or perpendicularly; as alſo they name thoſe lines GH, KN, *&c.* which are cut indifferently by the diameters, but at right angles by the Axis, thoſe, I ſay, they call *Ordinates*, or *Ordinate Applicates*, and their halves, FG, IK, *&c. Semiordinates*, (or ſome alſo call the latter *Ordinates*, and the former *double Ordinates*) and the parts of the Ax or Diameter EF, EI, *&c.* are called *Abſciſſa's* (by ſome *intercepted* axes and diameters)

Definition V.

PArticularly in the hyperbola they call the continuation of the ax or diameter ED 'till it meet the opposite side *c*B, *i. e.* to the vertex of the opposite section, the *Latus transversum* of the Hyperbola, to which there answers in the Ellipsis the axis or longest diameter, and so by latter Authors is called by the same name, but by *Apollonius* the *transverse Ax* or the *transverse Diameter*, as also the shortest ax or diameter is called the *Conjugate*, and its middle point O is called the *Center of the Section* or of the *opposite Sections.*

Definition VI.

THey called also a certain line EL (*Fig.* 101.) by the name of *Latus Rectum*, which is particularly to be found in all the sections, as we shall hereafter shew: And because this *Latus Rectum* is a sort of a Rule or Measure, according to which the squares or powers of the ordinates used to be estimated or valued (as we will shew in its proper place) therefore the Ancients used to call it by a Periphrasis *Linea secundum quam possunt Ordinatim applicatæ*, or the measure of the powers of the Ordinates; by some latter Authors it is called the *Parameter*. Now a mean proportional PQ found between this *Latus Rectum* and the *Latus Transversum* (*Fig.* 104 *n.* 1. and 2) (see also hereafter *Consect.* 2. *Prop.* 8.) and drawn thro' the centre O parallel to the Ordinates is called the *second Axis* or *Diameter*, or the *Conjugate* of the Hyperbola.

Definition VII.

NOW if we conceive the diameter or conjugate ax PQ brought down to the Hyperbola so that its middle point O shall touch its vertex in E, and from the center O you draw the right lines OR, OS, thro' the ends of this tangent line *p* and *q*, these are the lines which *Apollonius*, *Prop.* 1. *lib.* 2. demonstrates, that tho' by being continued, they always approach nearer and nearer to the curve GEH, and come so much

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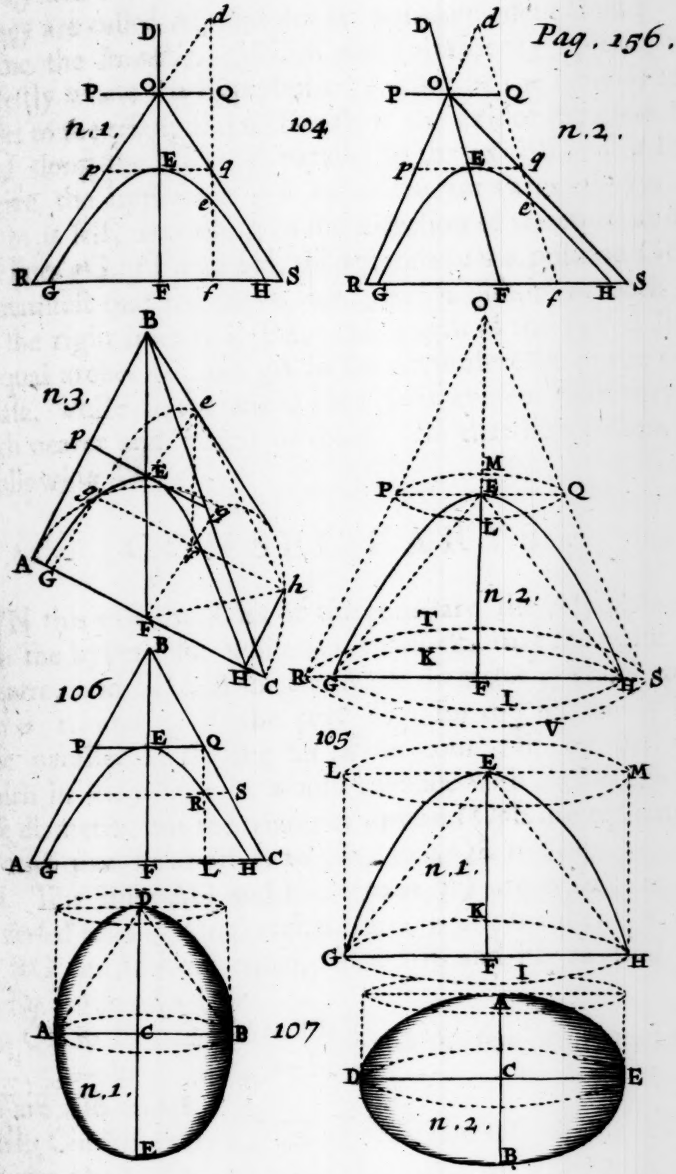
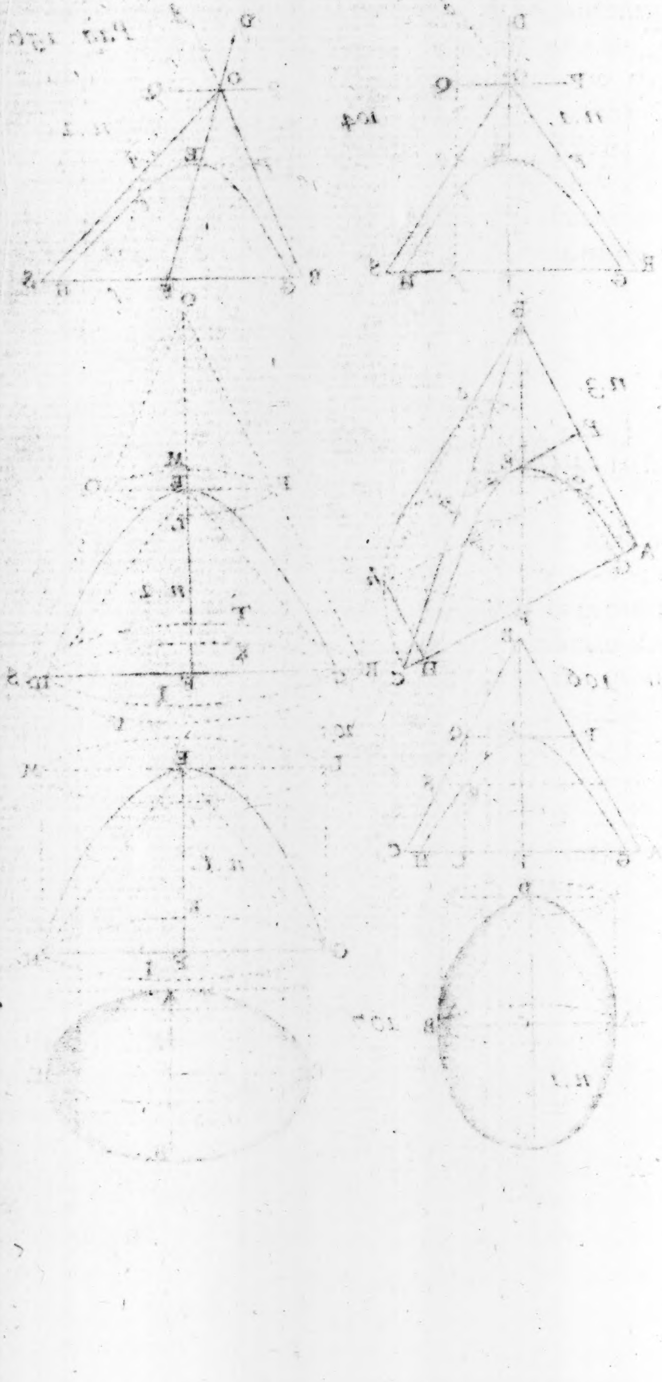




much the nearer by how much the farther they are continued, yet they will never concur with it or touch it, for which reaſon they are called Aſymptotes or non-coincident lines; and by ſome the *Intactæ*. Which non-coincidency appears moſt manifeſtly where the hyperbolical ſection of the cone is made parallel to the triangular ſection thro' the axis of the cone ABC (*n.* 3.) along the line *e. f.* parallel to the ax BF. For if we conceive the hyperbola *geh* to move forwards always parallel to it ſelf, according to the direction of the equal and parallel lines *g*G, *f*F, and *h*H, 'till it ſtands in the poſition GEH; it is manifeſt that the curve line GEH is diſtant on both ſides from the right lines BC, BA, the length of the verſed ſine of the equal arches *h*C and *g*A in the circumference of the circular baſe, while in the mean time it is evident that they approach nearer and nearer to them. So that hence there flow the following

CONSECTARYS.

I. IN this caſe the ſides of the cone are the Aſymptotes of the hyperbola, while it is manifeſt, that the point B is its centre, and EB half the tranſverſe diameter; which appears from *n* 1. and 2. of the preſ. *Fig.* for the ſection *ef* being made parallel to the the ax of the cone DF by *def.* 5. *d e* (which in the caſe *n.* 3. would coincide with *dq*) is the tranſverſe diameter, but the triangles *dpq* and O*p*E are equiangular, and conſequently as *p*E is to $\frac{1}{2}pq$, ſo is OE to $\frac{1}{2}dq$.

II. The lines AG and HC (*num.* 3.) are equal, as being the verſed ſines of equal arches; and in like manner (*n.* 1. and 2.) RG and HS are equal, ſince FR and FS as well as E*p* and E*q* are ſo alſo (for

$$\left.\begin{array}{l}\text{as OE to E}p\text{, ſo OF to FR}\\ \qquad\text{—E}q\qquad\qquad\text{—FS}\end{array}\right\}\text{and the ſemiordinates FG and}$$

FH are alſo equal.

III. Conſequently □ □ of RG into GS and of HS into HR are equal, &c. all which hereafter we will more univerſally demonſtrate.

Definition VIII.

(*a*) *Archim. de Conoid. & Spher. Def.* 1.

IF a Parabolick plane (*a*) HEGFH (*Fig.* 105. *n.* 1.) together with a triangle HEG inscribed in it, and a rectangle HL circumscribed about it, be conceived to be moved round about the ax EF on the point F; it will be evident that by the triangle there will be generated a cone, by the rectangle a cylinder, and by the parabola a parabolick solid, which with the comprehended cone, and the comprehending cylinder, will have the common base HIGK and the same altitude EF, and was by *Archimedes* named a *Parabolick Conoid.*

Definition IX.

(*a*) *Archim. de Conoid. & Spher. Def.* 3.

IF moreover an (*a*) hyperbolick plane HEGFH (*n.* 2.) with the inscribed triangle HEG, and another circumscribed one ROS made by the Asymptotes OR, OS, be conceived to be turned about the common ax OEF on the point F; it will be evident that there will be described by the inscribed triangle a cone comprehended within side, and by the hyperbola an *Hyperbolical Conoid* upon the same base HIGK and of the same heighth EF; and by the $\triangle$ ROS another cone which *Archimedes* calls the comprehending cone, whose base is RTSV, and its altitude composed of the axis of the hyperbola EF and half the transverse ax OE (which *Archimedes* called the additament of the ax of the hyperbola) and which we may commodiously divide into two parts, *viz.* into the cone OPMQL, whose base has for its diameter the conjugate ax PQ, and its altitude equal to half the transverse ax; and into a *Curti-cone* or detruncated cone terminated by the two bases PMQL and RTSV, but answering in altitude to the conoid and inscribed cone: From which, as comprehending it, if you take away the included conoid, there will remain the hollow curticone terminated below by the *Annulus* or ring RGIVHSKT and above by the circular base PMQL, and generated in the circumvolution by the intermediate lines EP, GR, *&c.* or the mixtilinear plane EGRP. Now if we suppose instead of this

this comprehending cone a circumſcribed cylinder on the ſame baſe and of the ſame altitude with the conoid and included cone, you'l have every thing like as in *Def.* 8.

CONSECTARYS.

NOW if we ſuppoſe the 1. caſe of *Def.* 9. to be expreſſed by the *Fig.* of *Def.* 7. *n.* 3. and conceive the preſent figure brought thence to be turned round about the ax BEF (*Fig.* 106.) we may deduce theſe following things in the room of Conſectarys for the foundation of our future demonſtrations.

I. The lines EQ, RS, HC, *&c.* of the mixtilinear ſpace comprehended between the hyperbola and the Aſymptotes (*viz.* the exceſs of the ordinates) altho' they are unequal, and by deſcending always grow leſs, yet in this circumvolution they will deſcribe equal circular ſpaces, *viz.* EQ a whole circle, (or circular plane) but RS and HC, *&c.* circular *Annuli* or rings all of the ſame bigneſs; which will thus appear to any one who compares this figure with the former: Since the ſpaces generated by the lines EQ, FC, *&c.* are as the ſquares of thoſe lines, and the □ of F*b* or FC exceeds the ſquare of *fb* or FH by the exceſs of the ſquare F*f* or E*e* or EQ, conſequently the quantity generated by FC will exceed that generated by FH the exceſs of that generated by EQ; and alſo that generated by FH by the exceſs of that generated by HC; it is manifeſt that the circle generated by EQ will be equal to the annular ſpace generated by HC; and the ſame will in like manner be evident of any ſpaces produced by RS.

II. Therefore the hollow detruncated cone generated by the ſpace EHCQ according to *Def.* 9. will be equal to a cylinder generated by the rectangle FIQE; for all the indiviſibles of the one, are equal to all the indiviſibles of the other by *Conſect.* 1.

Defi-

Definition X.

(*a*) *Archim. de Conoid. & Spher. Def.* 6.

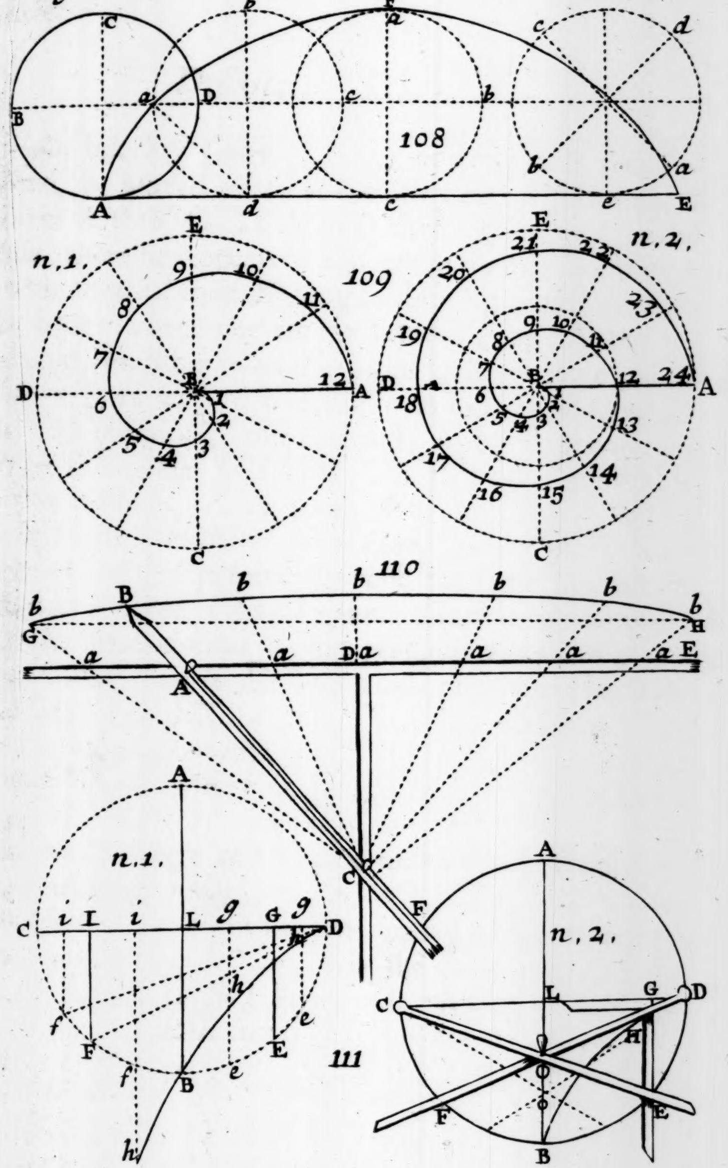
IF, laſtly, (*a*) an elliptical plane be turned about either of the axes, *viz.* either the longeſt DE (*Fig.* 107. *n.* 1) or the ſhorter AB (*n.* 2.) there will be thence formed an elliptical ſolid, called by *Archimedes* a ſpheroid; which in the firſt caſe will be an oblong or erect one, in the other a flat or depreſſed one: And it is ſelf evident, that if before this circumvolution of the ellipſis, there be inſcribed in one of its halves a triangle, and alſo a rectangle circumſcribed about it, having the ſame altitudes and baſes with the ſemi ellipſe, there will afterwards in the circumvolution be deſcribed by the triangle a cone, by the rectangle a cylinder, to which afterwards we will alſo compare the half ſpheroid; as alſo both the conoids with their inſcribed and circumſcribed cones and cylinders.

Definition XI.

IF upon the right line AE (*Fig.* 18.) you conceive a wheel or circle to rowl, until its point A, with which it touches the ſaid line, come to touch it again in E; the circle wil meaſure out the line AE equal to its periphery; but the point A by its motion will deſcribe the curve line AFE, which is called a *Trochoid* or *Cycloid*; and the area which this curve with the ſubtenſe AE comprehends, is named the *Cycloidal Space*; and the circle by whoſe motion they are determined is called the *generating Circle*.

CONSECTARY.

IT is evident from the geneſis of this curve that the deſcribing point *a* will always be as much diſtant in the circle from the point of contact *d* or *c*, as the point A in the right line paſſed over AE, is from the ſame point of contact, *i. e.* if the point *d* is diſtant from A the fourth part of the line AE, the arch *da* will alſo be the fourth part of the circle conſidered in this ſecond ſtation; and the point *c* being diſtant from A half of the interval AE, the arch *ca* will be alſo half of the circle,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circle, and ſo the point *a* coincide in the curve with F: And when the point *e* is diſtant from E only an eighth part of the whole line AE, the arch *ea* will alſo be the eighth part of the whole circle.

Definition XII.

If the right line BA (*num.* 1. 109.) one end at B remaining fixed, be moved round at the other end with an equal motion from A thro' C, D, E to A back again, and in the mean while, there be conceived another moveable point in it to move with an equal motion along the line BA from B to A, ſo that in the ſame moment wherein the moveable point A abſolves one revolution, the other moveable point ſhall alſo have paſſed thro' its right lined way, coinciding with the point A returned to its firſt ſituation; this extremity A by its revolution will deſcribe the circle ACDEA, and that other moveable point another curve B, 1, 2, 3, 4, *&c.* which with *Archimedes* we will call a *Helix* or *ſpiral Line*, and the plane ſpace comprehended under this ſpiral line and the right line BA in the firſt ſtation is called a *ſpiral ſpace.* Now if we ſuppoſe, *e.g.* the right lined motion of the point moving along BA to be twice ſlower than in the former caſe, ſo that (ſee *num.* 2.) in the ſame time that the point A makes one whole Revolution, the other moveable point ſhall come to F, making half the way BA, and then at length ſhall concur or meet with the extremity, when that ſhall have performed the other revolution; and ſo there will be deſcribed a double ſpiral line, whoſe parts with *Archimedes* we will ſo diſtinguiſh, that as he calls the part of the right line BF, paſſed over in the firſt revolution, ſimply the *firſt line*, and the circle made by the right line BF the *firſt Circle*; ſo we will call that part of the curve which is deſcribed in that time or revolution B 1 3 6 9 12 the *firſt Helix* or the *firſt Spiral*, and the area comprehended under it the *firſt ſpiral ſpace*: And, as the other part of the right line FA paſſed over in the other revolution is called the *ſecond line*, and the circle marked out by the whole line BA the *ſecond Circle*; ſo the curve deſcribed in the mean while 12, 15, 18, 24, may be called the *ſecond ſpiral line*, and the ſpace compre-

comprehended under it the *second spiral space,* and so onwards From these Definitions there flow the following

CONSECTARYS.

I. THE lines B 12, B 11, B 10, *&c.* drawn out, and making equal angles to the first or second spiral (and after the same manner (*a*) B 12, B 10, B 8, *&c.* or B 12, B 9, B 6, *&c.*) are arithmetically proportional, as is evident.

(*a*) *Archim. Prop.* 2. *of Spirals.*

II. The lines drawn out to the first spiral as B 7, B 10, *&c.* are one among another as the arches of the circles intercepted between BA and the said lines (*β*) B 7, B 10, *&c.* which is also evident to any one who considers what we did suppose; for in the same time as the end A passes over seven parts of the circle, the other moveable point will also run over seven parts of the right line BA, *&c.*

(*β*) *Archim. Prop.* 4.

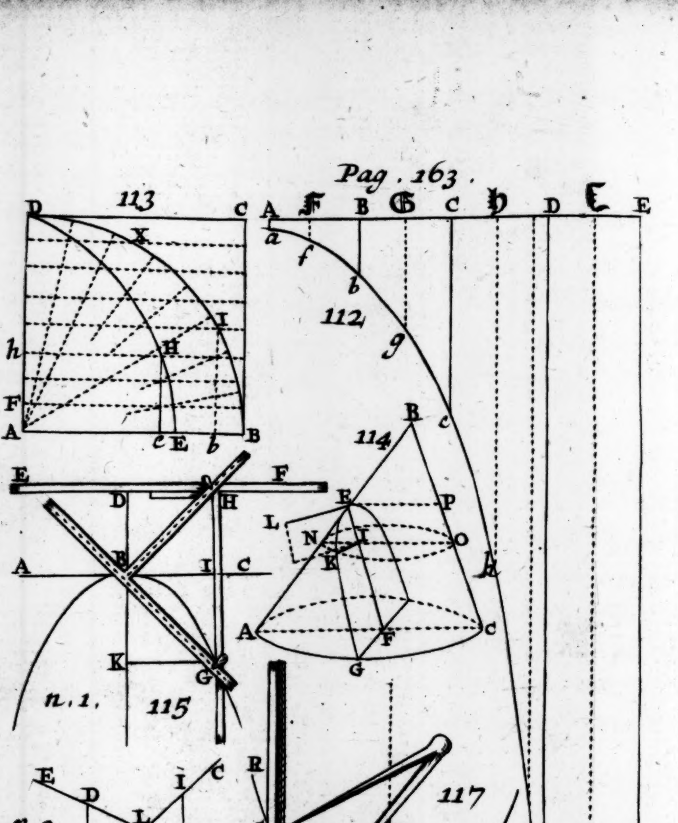
III. Lastly, The right lines drawn from the initial point (*γ*) B to the second spiral *e. g.* B 19 and B 22 (*num.* 2.) will be one to another as the aforesaid arches together with the whole periphery added to both sides: for at the same time the extremity A runs thro' the whole circle or 12 parts and moreover 7 parts (*i. e.* in all 19 parts) in the same time the other moveable point passes through 12 parts of the right line BA (in this case divided into 24 parts) and moreover 7 parts, that is, in all 19; and so in the others.

(*γ*) *Archim. Prop.* 15.

Definition XIII.

IF a right line BAF be conceived so to move within the right angles ADC, CDE, that on the one hand a certain point C fixed in one leg of the *Norma* or ruler may always glide along, and on the other hand a certain moveable point A may always run along the other side of the *Norma* (which complicated motion of the describing rule BAF, after what way it may be organically procured, may be seen by the

Pag. 163.



the 110*th*. Figure;) by the extreme point of the moveable line B there will be deſcribed a curve called by its Inventor *Nicomedes* the *firſt Conchoid*, whereof this is a property, that the right lines CB, C*b*, drawn from its centre C to its *Ambitus* or curvity are not themſelves, as in the circle, equal, but yet have all the parts, AB, *ab*, intercepted between the curve, and the directrix AE are equal; as is evident from its geneſis.

Definition XIV.

IF, the diameters of a circle being AB and CD (*Fig.* 111. *n.* 1.) cutting one another at right angles, you take BE or B*e* and BF or B*f* equal arches, and from E or *e* you draw the perpendiculars EG or E*g*, and through theſe from D to F or *f* you draw the tranſverſe lines DF or D*f*; the ſeveral points of interſection H, *h*, *&c.* decently connected together will exhibit the curve line D*h*, H*h*B (which may be continued alſo without the circle if you pleaſe, and) which is commonly attributed to *Diocles*, and called a *Ciſſoid*.

Definition XV.

IF, having divided the right line AE (*Fig.* 112.) into the equal parts AB, BC, *&c.* from the points of the diviſion A, B, C, *&c.* you draw the parallel lines A*a*, B*b*, C*c*, *&c.* in geometrical progreſſion (as *e. g.* let A*a* be 1, B*b* 4, C*c* 16, D*d*, 64, *&c.* or B*b* 10, C*c* 100, D*d* 1000, E*e* 10000, *&c.*) and further biſecting AB, BC, *&c*, 𝔉, 𝔊, ℌ, ℑ, you let fall mean proportionals between the next Collaterals 𝔉𝔣, 𝔊𝔤, ℌ𝔥, ℑ𝔦, *&c.* and continue to do ſo till the parallels are brought near to one another; the curve line drawn thro' the extremities of theſe lines *a*𝔣, *b*𝔤, *c*𝔥, *d*𝔦, will be the logarithmical line of the moderns, whoſe properties and uſes are very excellent.

SCHOLIUM.

AMong thoſe uſes, that is none of the leaſt, from which this curve borrows its name, *viz.* in ſhewing the nature and invention of Logarithms. For, *e g.* 1. If this line was accurately delineated in a large ſpace, the parts AB,

BC, *&c.* being taken so big, that they might be subdivided not only into 100 or 1000 but into 10000 or 100000 parts; making AB 100000 (and so A 00000) AC would be 200000, AD 300000, *&c.* while in the mean while there answer to these as primary Logarithms in arithmetical progression the geometrical proportional numbers, A*a* 1, B*b* 10, C*c* 100, D*d* 1000, E*e* 10000, *&c.* Whence, 2. Its Logarithm may be assign'd to any given intermediate number, *e.g.* to the number 982, for having cut off this number from D*d* by a geometrical scale on the line DM, if you draw M*n* parallel to AD, and *n*N parallel to DM, it will give AN on the same scale, *viz.* the Logarithm sought, and reciprocally. But if, 3. it seem difficult to delineate a Figure so large, yet at least the clear conception of such a delineation evidently shews the arithmetical method, which those ingenious Men have made use of, who have made the tables of Logarithms with a great expence of Labour and pains, *viz.* by finding continual mean proportionals, arithmetical ones between any two Logarithms already known, and geometrical ones between two vulgar numbers answering to them, *&c.* by comparing what we have noted in *Schol.* 2. *Prop.* 20. *Lib.* 1. And we will note, 4 out of *Pardies*, that, since the Logarithms of numbers distant from one another by a decuple proportion, differ by the number 100000, having found the Logarithms of all the numbers from 1000 to 10000 you will at the same time have all the Logarithms of all the other numbers that are between 100 and 1000, between 10 and 100, and between 1 and 10, only changing the characteristick, and lessening it in the first case by unity, in the second by 2, in the third by 3; as *e.g.* if of the number 9, 900 you had found the Logarithm 399, 563, the Logarithm of the subdecuple number 990 would be (*viz.* substracting from the former 100000) 299, 563. and the Logarithm again of this subdecuple of this 99 would be 199, 563, *&c.* Thus in the Chiliads of *Briggs* to the number

99000	Answ.	4, 99563, 51946
9900	——	3, 99563, 51946
990	——	2, 99563, 51946
99	lastly ——	1, 99563, 51946.

But

But there will not ariſe ſuch advantage for making Logarithms by this obſervation as it may at firſt ſight ſeem to promiſe, becauſe there are 9000 numbers between 1000 and 10000 whoſe Logarithms muſt be found alſo, and but 900 between 100 and 1000, and but 90 between 10 and 100, and but 9 between 1 and 10, and ſo in all 999, which is not the ninth part of the former.

Definition XVI.

IF the radius AD (*Fig.* 113.) be conceived to move equally about the point A through the periphery of the quadrant DB, while in the mean time the ſide of the ſquare DC remaining always parallel to it ſelf, deſcends alſo with an equal motion thro' DA, ſo that in the ſame moment the radius AD and the aforeſaid ſide DC ſhall fall upon the baſe AB; or (if any one ſhould think that this way the proportion of a right line to a circular one is ſuppoſed by a ſort of *Petitio Principii* or begging the queſtion) the right line DA as well as the quadrant DB being divided into as many equal parts as you pleaſe (*e. g.* here both of them into 8) and drawing thro' theſe from the center A ſo many Radii and thro' them parallel lines; the points of interſection being orderly connected together will exhibit a curve line, whoſe invention is attributed to *Dinoſtratus* and *Nicomedes* in the fourth Book of *Pappus Alexandrinus*, and which from its uſe is called a *Quadratrix*, it having among the reſt this property, that from AB it cuts off a part AE, which is a third proportional to the quadrant DB and its radius DA; which hereafter we will demonſtrate. In the mean while from this deſcription of it, you have theſe

CONSECTARYS.

I. IF thro' any point H aſſumed in the *Quadratrix* you draw the radius AHI, and from the ſame point the perpendiculars H*h* and H*e*, the whole arch DB will always be to the part IB cut off, as the whole line DA to the part *h*A cut off, or H*e* equal to it.

II. Consequently therefore any given arch or angle of the quadrant *e.g.* IB or IAB may by help of the quadratrix be divided into three equal parts or as many as you please, or in what proportion soever you will; while having drawn the radius AI, the perpendicular H*a* let fall from the point of the quadratrix H, may be divided into three or as many equal parts as you please, or in any proportion whatsoever, and thro' these sections radius's drawn to divide the arch.

BOOK II.

SECTION II.

CHAP. I.

Of the chief Properties of the Conick Sections.

Proposition I.

(α) I. *Property of the Parab. Apoll. Prop.* 11. *Lib.* 1.

IN *the Parabola (GKEH Fig* 114) *the* (α) *square of the semi-ordinate (IK) is equal to the rectangle IL made by the Latus Rectum EL and the abscissa EI.*

Demonstration.

MAke the sides of the cone that is supposed to be cut, AB = *a*, BC = *b*, and moreover EB = *oa*, and EI = *eb*, and AC = *c*; therefore NI will be = *ec*, by reason of the similitude of the △ △ BCA and EIN; and EP or IO = *oc*, by reason of the similitude of the △ △ ABC and EBP. Therefore ▭ NIO = *oecc* = □ IK by the *Schol.* of *Prop.* 34 (*n.* 3) and *Prop* 17. *Lib.* 1. Now if a line be sought which with the abscissa EI shall make the ▭ IL = □ IK you will have

have it by dividing the said square by the Abscissa EI. *viz.* $\frac{occ}{eb}$ i. e. $\frac{occ}{b} = $ EL. And this is called the *Latus Rectum*, *viz.* in relation to the Abscissa EI with which it makes that rectangle, which, it's evident, is $= \square$ IK, and from this equality the section has the name of Parabola, in *Apollonius*.

CONSECTARYS.

I. THis *Latus Rectum*, expressed by the quantity $\frac{occ}{b}$, may be found out after a shorter way, if you make as *b* to *c* (the side of the cone parallel to the section BC at the Diameter of the base AC) so *oc* (the side EP called by some the *Latus Primarium*) to a fourth.

II. But if any one, with *Apollonius*, had rather express this by meer *data* in the cone it self as cut (because *oc* or that *Latus Primarium* EP is not a line belonging to the cone it self) he may easily perceive, if the quantity of the *Latus Rectum* found above, be multiplyed by the other side of the cone *a*, there will be produc'd the equivalent $\frac{oacc}{ab}$ which instead of the proportion above will furnish us with this other,

as *ab* —— to *cc* —— so *oa* } to a fourth;
$\square$ of AB into BC — $\square$ AC —— EB }

which is the very proportion of *Apollonius* in *Prop.* 9. *Lib.* 1. and confirms our former.

SCHOLIUM I.

HEnce you have an easie and plain way of describing a Parabola, having the top of the ax and the *Latus Rectum* given, *viz.* by drawing several semiordinates whose extreme points connected together will exhibit the curvity of the Parabola. But you may find as many semiordinates as you please, if having cut off as many parts of the Ax as you please, you find as many mean proportionals between the

Latus

Latus Rectum and each of those parts or Abscissa's. See *n.* 2. and 3. *Fig.* 47. *Introduct. to Specious Analysis.*

SCHOLIUM II.

HEnce also we have a new genesis of the parabola in *Plano* from the speculations of *De Witte*, *viz.* if the rectilinear angle HBG (*Fig.* 115.) conceived to be moveable about the fixed point B be conceived so to move out of its first situation with its other leg BH along the immoveable rule EF, that it may at the same time move also the ruler HG, from its first situation DK, all along parallel to it self, and with the other leg BG let it all along cut the said ruler HG, and with this point of its intersection continually moving from B towards G it will describe a curve. That this curve will be the parabola of the antients is hence manifest, because it will have this same first property of the parabola. For, 1. if the angle HBG (*n.* 1.) be supposed to be a right one, and BD or HI $= a$, BI or KG $= b$ (*viz.* in that station of the angle and rule HG by which they denote the point G in the intersection) you'l have by reason of the right angle at B, BI, *i. e.* b a mean proportional Between HI *i. e.* a, and IG or BK, and so this as an abscissa $= \frac{bb}{a}$. Wherefore if BK *i. e.* $\frac{bb}{a}$ be multiplyed by BD $= a$, the rectangle DBK will be $= bb = \square$ KG; which is the first property of the parabola: So that it follows, since the same inference may be made of any other point in this curve, that this curve will be the parabola, BD or HI its *Latus Rectum*, KG a semiordinate, and BK its axis, &c. 2. If the angle HBG be an oblique one (*num.* 2) it may be easily shewn from what we have supposed that the $\triangle\ \triangle$ DBH and BKG will be equiangular: Therefore as BD (i. e. a) to DH sc. BI (i. e. b) so KG sc BI (i. e. b) to BK (i. e. $\frac{bb}{a}$.) Therefore again the $\square$ DBK $= bb\ \square$ KG. Q E D.

Consect. 3. It is also evident in this second case, that BK drawn parallel to the ax, but not thro' the middle of the parabola

rabola, will be a diameter which will have for its vertex B, its *Latus Rectum* BD, and ſemiordinate GK, *&c.*

Conſect. 4. Therefore you may find the *Latus Rectum* in a given parabola geometrically, if you draw any ſemiordinate whatſoever IK (*Fig.* 116.) and make the abſciſſa EF equal to it, and from F draw a parallel to the ſemiordinate IK, and from E draw the right line EK thro' K cutting off FH the *Latus Rectum* ſought; ſince as EI to IK ſo is EF (*i. e.* IK) to FH by *Prop* 34. *lib.* 1. wherefore having the abſciſſa and ſemiordinate given arithmetically, the *Latus Rectum* will be a third proportional.

Conſect. 5. Since therefore the *Latus Rectum* found above is $\frac{occ}{b}$, if you conceive it to be applyed to the parabola in LM, ſo that N ſhall be that point which is called the *Focus*, LN will be $\frac{occ}{2b}$ and its ſquare $\frac{ooc^4}{4bb}$ and this divided by the *Latus Rectum* $\frac{occ}{b}$ will give $\frac{occ}{4b}$ for the abſciſſa EN: So that the *diſtance of the Focus from the Vertex* will be $\frac{1}{4}$ of the *Latus Rectum.*

Conſect. 6. Since therefore EN is $= \frac{occ}{4b}$, if for EF you put ib, NF will be $= \frac{ib - occ}{4b}$, whoſe ſquare will be found to be $i^2b^2 - \frac{oicc}{2} + \frac{o^2c^4}{16b^2}$ to which if there be added $\square$ GF $= oicc$, by *Prop.* 1. the ſquare of NG will be $= i^2b^2 + \frac{oicc}{2} + \frac{o^2c^4}{16b^2}$ whoſe root (as the extraction of it and, without that, the analogy of the ſquare NF with the ſquare NG manifeſtly ſhews) will be $ib + \frac{occ}{4b}$; ſo that a right line drawn from the *Focus* to the end of the ordinate, will always be equal to the abſciſſa EF + EN

† EN *i. e.* (if EO be made equal to EN) to the compounded line FO.

SCHOLIUM III.

HEnce you have an easier way of describing the parabola in *Plano* from the given *Focus* and *Vertex*, *viz.* (*Fig.* 117.) the axis being prolonged thro' the vertex E to O, so that EO shall ⵌ EN, if a ruler HI be so moved by the hand G, according to PQ, from OF to HI, that putting in a style or pin, it shall always keep the part of the Thred NGI (which must be of the same length with the rule HI) as fast as if it were glued to it (which perhaps might also be done with the Compasses by an artifice which we will hereafter also accommodate to the hyperbola) and at the same time it will describe in *Plano* the part of the line EGR. That this will be a parabola is evident from the foregoing Consect. because as the whole thred is always ⵌ to the ruler IH; so the part GN is always necessarily equal to the part GH, *i. e.* to the line FO. Moreover from the same sixth Consect. and *Fig.* 116. may be drawn another easie way of describing the parabola in *Plano* from the *Focus* and Vertex given thro' innumerable points G to be found after the same way: *viz.* If from any assumed point in the ax F you draw to the ax a perpendicular, and at the interval FO from the *Focus* N you make an intersection in G. Which innumerable points G will be determined with the same facility, having given only, or assumed the axis and *Latus Rectum*, by vertue of the present Proposition. For if, having assumed at pleasure the point F in the axis, you find a mean proportional between the *Latus Rectum* and the abscissa EF, a semiordinate FG made equal to it, will denote or mark the point G in the parabola sought.

Propo-

Proposition II.

In the hyperbola (GKEH *Fig* 118) (α) *the square of the semiordinate* (IK) *is equal to the rectangle* (IL) *made of the Latus Rectum* (EL) *and the abscissa* (EI) *together with another rectangle* LS *of the said abscissa* (EI *or* LR) *and* RS *a fourth proportional to* DE *the Latus Transversum,* (EL) *the Latus Rectum, and* EI *the Abscissa.*

(α) I. *Property of the Hyperb. Apoll. Prop.* 12. *Lib.* 1.

Demonstration.

Suppose the side of the cone AB here also $= a$, and BM parallel to the section $= b$, and the intercepted line AM $= c$, and EI $= eb$; all according to the analogy we have observed in the parabola; and NI will be as there $= ec$. Making moreover MC $= d$ and the *Latus Transversum* DE $= ob$, so that DI shall be $= ob + eb$; then will (by reason of the similitude of the △ △ BMC, DEP, and DIO) EP be $= od$, and IO $= od + ed$, and so QO $= ed$ Therefore □ NIO will be $= oecd + eecd =$ □ IK But this square divided by the Abscissa EI $= eb$ gives $\frac{oecd + eecd}{eb}$ or $\frac{ocd + ecd}{b}$ for the line IS which with the abscissa would make the rectangle ES = to the said square IK. Now therefore, if here also we call a Line the *Latus Rectum* found after the same way as in the parabola, *viz.* by making

as b —— to c —— so od to a fourth $\frac{odc}{b}$

(as a line parallel to the section — to the intercepted diam. so the *Latus Primarium*, but that the other part $\frac{ecd}{b}$ will be a fourth proportional to bc and ed or to eb, ec and ed, or (to speak with *Apollonius* as we have done in the Prop.) to ob, $\frac{ecd}{b}$ and eb (for in these three cases you l have the same fourth $\frac{ecd}{b}$.) Wherefore now it is evident that the square of the se-

miordinate *oecd* † *eecd* is equal to the rectangle IL (made by the *Latus Rectum* $\frac{ocd}{b}$ into the abscissa *eb* = *oecd*) together with ▭ LS of this fourth proportional $\frac{ecd}{b}$ into the same abscissa E*b*, which is = *eecd*. Which was to be found and demonstrated.

CONSECTARYS.

I. Hence you have in the first place the reason why *Apollonius* called this Section an Hyperbola; *viz.* because the square of the ordinate IK exceeds or is greater than the rectangle of the *Latus Rectum* and the Abscissa.

II. Since therefore the *Latus Rectum* here also as well as in the parabola is found by making as *b* to *c* so *od* to $\frac{odc}{b}$ (*i.e.* as the parallel to the section BM is to the intercepted Diam. AM so is the *Latus Primarium* EP to a fourth EL.) If any one had rather express this *Latus Rectum* after *Apollonius*'s way, he will easily perceive, this quantity being found and multiplyed both Numerator and Denominator by *b* the parallel to the section, there will come out the equivalent quantity $\frac{obcd}{bb}$ which gives us instead of the former proportion this other,

as *bb* — to *cd* — so *ob*
□ BM — ▭ AMC — *Latus Transversum* } to a fourth;

which is that of *Apollonius* in *Prop.* 12. *Lib.* 1. and consequently herein confirms our former.

III. You may also have this *Latus Rectum* geometrically, by finding a third proportional (as we have done in the parabola *Consect.* 4. *Prop.* 1.) to the abscissa EI (*Fig.* 119.) and the semiordinate IK (= EF;) and then find a fourth proportional EL to DI (the sum of the *Latus Transversum* and abscissa) and FH already found, or IS equal to it, and DE (the *Latus Transversum*) and that will be the *Latus Rectum* sought.

SCHO-

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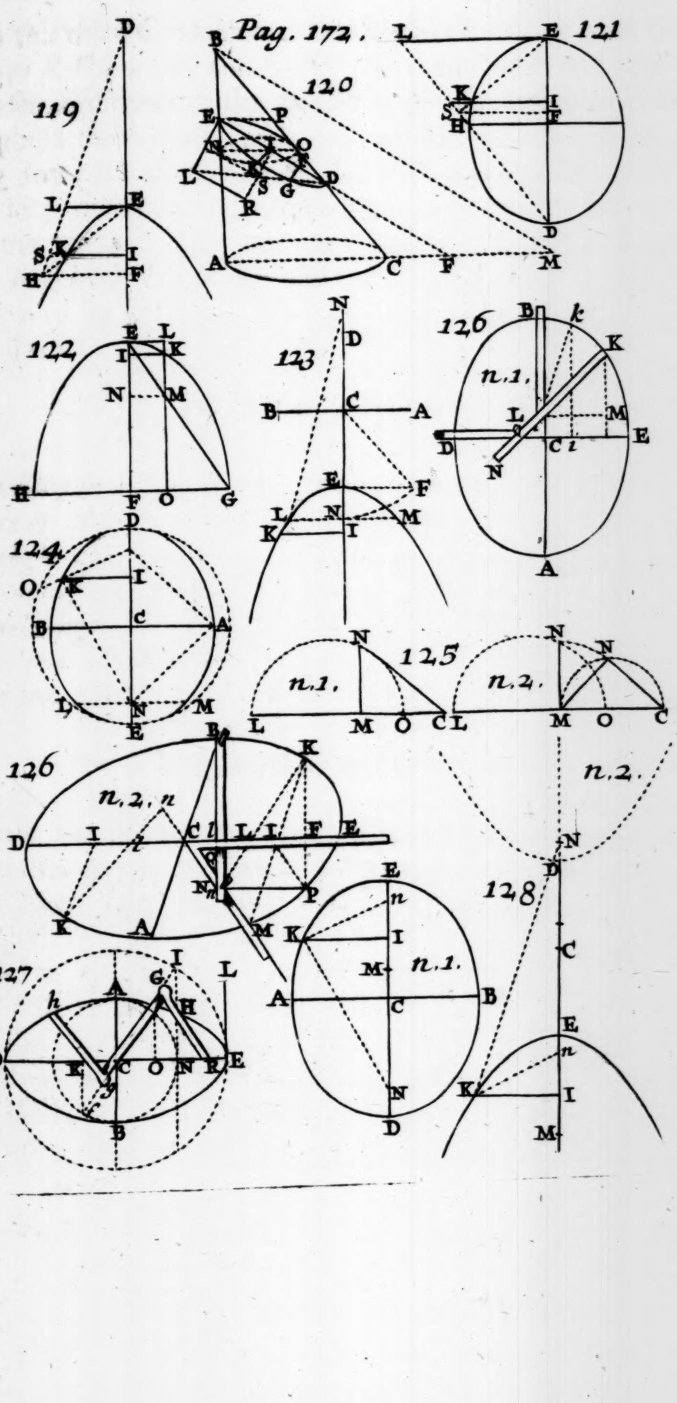


SHCOLIUM.

FRom this third Consectary, we may reciprocally from the *Latus Rectum* and transverse given, find out and apply as many semiordinates to the ax as you please, and so describe the hyperbola thro' their (ends or) infinite points: *viz.* if assuming any part of the abscissa EI, you make as DE to EL so DI to IS; and then find a mean proportional IK between IS and the abscissa EI, and that will be the semiordinate sought: And both this praxis and the Consect. may be abundantly proved by setting it down in, and making use of, the literal *Calculus.*

Proposition III.

IN the Ellipsis (KDEK, Fig. 120.) the (a) square of the semiordinate (IK) is equal to the rectangle (IL) of the Latus Rectum (EL) and the abscissa (EI) (less or) taking first out another rectangle (LS) of the same abscissa (EI or LR) and RS a fourth proportional to (DE) the Latus Transversum (EL) the Latus Rectum and (EI) the abscissa.

(a) I. *Propert. of the Ellipse of Apoll. Prop.* 13. *Lib.* I.

Demonstration.

Suppose the side of the cone to be AB here also = *a* and BM parallel to the section = *b* and the intercepted AM = *c*, and EI = *eb*; and NI will be again = *ec*, all as in the hyperbola. And making also here as in the hyperbola MC = *d*, and the *Latus Transversum* DE = *ob*, so that DI will be *ob* — *eb*; then will (by reason of the similitude of the △ △ BMC, DEP and DIO) EP be = *od*, and IO = *od* — *ed*. Therefore $\square$ of NIO will be = *oecd* — *eecd* = □ IK. But this square divided by the abscissa EI = *eb* gives $\frac{oecd - eecd}{eb}$ or $\frac{ocd - ecd}{b}$ for that line IS which with the abscissa would make the rectangle ES = to the said square IK. Now therefore if we call the *Latus Rectum* a right line found after the same way as in the parabola, by making according to *Cons.* I. *Prop.* I.

as b to c —— so od — to a fourth $\frac{ocd}{b}$ *i. e.* as the line parallel to the section —— to the intercepted diameter —— so the *Latus Primarium*, &c. It is manifest that the *Latus Rectum* is one part of the line just now found; and the other part $\frac{ecd}{b}$ is a fourth proportional to b, c and ed, or (to speak with *Apollonius* as we have done in the Prop.) to ob, $\frac{ocd}{b}$ and eb (for there will come out the same quantity $\frac{ecd}{b}$;) wherefore now it is evident that the □ of the semiordinate IK is equal to the ▭ IL (of the *Latus Rectum* $\frac{ocd}{b}$ and the abscissa eb = $oecd$) having first taken out thence the ▭ LS, or $eecd$ out of that fourth proportional $\frac{ecd}{b}$ by the same abscissa eb; which was to be found and demonstrated.

CONSECTARYS.

I. Hence you have first of all the reason of the name of the *Ellipse*, which *Apollonius* gave to this section; *viz.* because the square of the semiordinate IK is defective of, or less than the rectangle of the *Latus Rectum* and the abscissa.

II. Since therefore the *Latus Rectum* here also as well as in the parabola and hyperbola, is found by making as b to c so od to $\frac{ocd}{b}$ (*i. e.* as BM parallel to the section is to the intercept. diam. AM so the *Latus Primarium* EP to a fourth EL) now if any one had rather express this *Latus Rectum* after *Apollonius*'s way, he will easily see that the quantity above found being multiplyed both Numerator and Denominator by b, that there will come out an equivalent one $\frac{obcd}{bb}$, which instead of the former proportion will give this other,

$$\left.\begin{array}{l} \text{as } bb \text{ — to } cd \text{ — so } ob \\ \text{BM — } \square \text{ AMC — } \textit{Latus Transvers.} \end{array}\right\} \text{to a fourth;}$$

which is the same with that we have also found in the hyperbola, and which also *Apollonius* has *Prop.* 13. *Lib.* 1.

III. This *Latus Rectum* may also be had geometrically, if you find, 1. in the hyperbola a third proportional FH to the abscissa EI (*Fig.* 121.) and semiordinate IK (= EF.) 2. But EL a fourth proportional to DI (the difference of the *Latus Transversum* and the abscissa) and the found FH, or IS equal to it, and the *Latus Transversum* DE, is the *Latus Rectum* sought.

SCHOLIUM.

FRom this third Consect. we may reciprocally, having the *Latus Rectum* and *Transversum* given, apply as many semiordinates to the ax as you please, and so draw the ellipsis thro' as many points given as you please, *viz.* if, taking any abscissa EI, you make as DE to EL so DI to a fourth IS; then between this IS and the abscissa EI find a mean proportional IK, and that will be the semiordinate sought: And this *Praxis* also and the third Consect. may be abundantly proved by making use of a literal *Calculus*. For *e. g.* here a fourth proportional to ob, $\frac{ocd}{b}$ and $ob - eb$ will be $\frac{ocd - ecd}{b}$; and a mean proportional between this fourth and eb will be $\sqrt{oecd - eecd}$, &c.

Proposition IV.

IN a Parabola (α) *the squares of the ordinates are to one another as the abscissa's.*

(α) 2. *Property of the Parab.* 20. *Prop. Apoll. Lib.* 1. *Con.*)

Demon-

Demonstration.

For if EF (*Fig*. 122.) be called *ib*, as above EI was called *eb*, since the *Latus Rectum* is $\frac{occ}{b}$; the square of FG will be $=$ *oicc*. Therefore

$$\left.\begin{array}{l}\square\ \text{IK will be to}\ \square\ \text{FG as} \\ oecc \text{ --------- } oicc\end{array}\right\}\begin{array}{l}e \text{ to } i \text{ or} \\ eb \text{ to } ib.\end{array} \quad \text{Q. E. D.}$$

CONSECTARY.

HEence having drawn LO parallel to the ax or diameter EF, if it be cut by the transverse line EG in M and by the curve of the parabola in K; then will OL, ML, and KL be continual proportionals. For EF is to EN as FG to NM or IK, by reason of the similitude of the △ △ EFG and ENM. But the squares FG and IK are in duplicate proportion of EF to EN by *Prop*. 35. *Lib*. 1. and are also in the same proportion as the abscissa's EF and EI by the pres. Therefore EF to EI is also in duplicate proportion of EF to EN *i. e.*

$$\left\{\begin{array}{l}EF \\ OL\end{array}\right. \text{is to} \left.\begin{array}{l}EN \\ ML\end{array}\right\} \text{as} \left.\begin{array}{l}EN \\ ML\end{array}\right\} \text{to} \left.\begin{array}{l}EI \\ KL\end{array}\right\} \text{Q. E. D.}$$

Proposition V.

(α) *Property of the Hyperbola and Ellips. Apol*. 21. *Lib.* 1.

IN the hyperbola and Ellipsis (α) the squares of the Ordinates are as the rectangles contained under the lines which are intercepted between them, and the Vertex's of the Latus Transversum's.

Demonstration.

For, if EF (*Fig*. 118. and 120.) be called *ib*, as EI was above called *eb*, then will according to *Prop*. 2. and the 3*d*. deduction.

$$\begin{array}{lll} GF = & oicd + eicd & \text{in the Hyperb.} \\ & oicd - eicd & \text{in the Ellips.} \\ \text{and the} \ \square\ \text{DFE} = & oibb + iibb & \text{in the Hyperb.} \\ & oibb - iibb & \text{in the Ellips.} \end{array}$$

There-

Therefore the □ KI is to the ſquare GF as $oecd \pm eecd$ to $oicd \pm iicd$ i.e. as $oe \dagger ee$ to $oi \dagger ii$.

and ▭ DIE is to the ▭ DFE, as $oebb \pm eebb$ to $oibb \pm iibb$ i. e. in like manner as
$oe \pm ee$ to $oi \pm ii$. Q. E. D.

CONSECTARY I.

IN the Ellipſis this may be more commodiouſly expreſſed apart thus; the ſquares of the ordinates (KI and GF) are as the rectangles contained under the ſegments of the Diameter (*viz.* DIE and DFE) in which ſenſe this property is alſo common to the circle, as in which the ſquares of the ordinates are always equal to the rectangles of the ſegments.

CONSECTARY II.

THerefore, if the *Latus Rectum* be conceived to be applyed in the hyperbola, ſo that N ſhall be the *Focus*; (ſee *Fig.* 123.) then will LN $= \frac{ocd}{2b}$, and its ſquare be $\frac{ooccdd}{4bb}$. But as the □ KI to the ſquare LN, ſo is the ▭ DIE to the ▭ DNE i. e. $oecd \dagger eecd$ to $\frac{ooccdd}{4bb}$ ſo is $oebb \dagger eebb$ to $\frac{oocd}{4}$. But now the ▭ of the whole DE and the part added EN into the part added EN, *i. e.* ▭ DNE ($= \frac{oocd}{4}$ together with the ſquare of half CE ($= \frac{oobb}{4}$) is $=$ □ compounded of half and the part added CN $= \frac{oocd \dagger oobb}{4}$ by *Prop.* 9. *lib.* 1. Wherefore CN the diſtance of the *Focus* from the centre is $= \sqrt{\frac{oodc}{4} \dagger \frac{oobb}{4}}$. But $\frac{oocd}{4}$ is the fourth part of the ▭ of the *Latus Tranſverſum* ob and the *Latus Rectum* $\frac{ocd}{b}$ (or the fourth

fourth part of the figure, as *Apollonius* calls it) and $\frac{oobb}{4}$ is the □ of $\frac{ob}{2}$ i. e. of half the *Latus Transversum.* Wherefore we have found the following Rule of determining the *Focus* in an hyperbola: *If a fourth part of the figure* (or the rectangle of the *Latus Rectum* into the *Transversum) be added to the square of half the* Latus Transversum, *and from the sum you extract the square root; that will be the distance of the Focus from the center CN: And hence substracting half the* Latus Transversum CE, *you will have distance of the Focus from the Vertex EN.*

CONSECTARY III.

IN like manner in the Ellipsis having drawn the ordinate LM (*Fig.* 124.) that the *Focus* may be in N, the □ LN would be $\frac{ooccdd}{4bb}$ as above, and by a like inference □ DNE = $\frac{oocd}{4}$. But now □ DNE together with the square of the difference CN is equal to the □ of half CE by *Propos.* 8. *lib.* 1. and consequently the □ CN is = □ CE —— □ DNE, that is, $\frac{oobb}{4} - \frac{oocd}{4}$. Wherefore CN the distance of the *Focus* from the centre is = $\sqrt{\frac{oobb - oocd}{4}}$. Wherefore we have found the following Rule to determine the *Focus* in the Ellipse. *If the fourth part of the figure* (or the rectangle of the *Latus Rectum* into the *Latus Transversum) be substracted from the square of half the* Latus Transversum, *and from the remainder you substract the square root; that will be the distance of the Focus from the Centre CN: And taking hence half the* Latus Transversum *CE, you'l have the distance of the Focus from the Vertex EN.*

SCHO-

SCHOLIUM I.

Both the Rules are easie in the practice, for since $\frac{oobb}{4}$ is nothing but the square of CE, and $\frac{oocd}{4}$ nothing but the rectangle of $\frac{1}{4}$ DE into LM; if between LM and $\frac{1}{4}$ DE or MO (*Fig.* 125.) you find a mean proportional MN, (and so whose □ is equal to that ▭) and in the hyperbola join to it at right angles MC = CE, the hypothenusa CN will be the distance sought of the *Focus* from the centre: And the same may be had in the Ellipsis, if (*n.* 2.) having described a semi-circle upon CM = CE you draw or apply the mean found MN, and draw CN.

SCHOLIUM II.

Hence also we have (*a*) a *new genesis of the Ellipse* in *Plano* about the diameters given, from the speculations of Monsieur *de Witt*; *viz.* If about the rectilinear angle DCB (*Fig.* 126. *n.* 1. and 2.) consider'd as immoveable, the rule NLK (which all of it will equal the greater semidiameter CB, and with the prominent part LK the lesser CD) be so moved that N going from C to D, and L from B to C may perpetually glide along the sides of the angle, the extreme point in the K in the mean while describing the curve BKE (and in a like application the other quadrants) and that this curve thus described will be the ellipsis of the ancients is hence manifest, because it has the second property of the ellipse just now described. For, 1. if the angle DCB or NCB be supposed to be a right one (as in *Fig.* 126. *num.* 1.) and the rule KN in the same station, it marked out the point K, and having apply'd the semiordinate KI, and drawn the perpendicular LM, from the square KL and the square CE (as being equal) substract mentally the equal squares LM and CI, and there will remain by virtue of the Pythagorick Theor. on the one hand □KM and on the other by

(*a*) *De Witt Elem. Curv. Lib.* 1. *Cap.* 3. *Prop.* 13.

Prop. 8. *lib.* 1. ▭ DIE equal among themselves. But now the square of KI is to the square of KM (*i e.* to the ▭ DIE) as the square of KN to the square of KL (*i. e.* as the square of CB to the square of CE) by reason of the similitude of the △ △ KLM and KNI; and since the same may be demonstrated after the same manner of any other semiordinate K*i* *viz.* that its □ K*i* is to the ▭ D*i*E as the square CB to the square CE. It also follows, that the □ KI is to the ▭ DIE as □ K*i* to the ▭ D*i*E, and alternatively, the square KI will be to the square K*i* as the ▭ DIE to the ▭ D*i*E; which is the second property of the ellipse. 2. If the angle NCE be not a right one (as in *Fig.* 126.*n.*2. and the like cases) having drawn NO and KP parallel to the rule *nl*B in the first station, [in which station the angle NCE, to which the flexible ruler is to be made, is determined, *viz.* by letting fall the perpendicular B*l* from the extremity of one diameter upon the other, and moreover by adding or substracting the difference of the semi-diameters *ln*] having also drawn the Ordinate KIM, and PI parallel to CN; which being done the △ △ CB*l* and IKF, and also CB*n* and IKP will be similar. Wherefore having joined NP, from the parallelism of the lines IP and NC and the similitude of the aforesoid △ △, as also of NCO and *n*C*l*, it will be easie to conclude that NCIP is a parallelogram. Wherefore since KN is = CE and □ KN = □ CE, having substracted the squares of the equal lines NP and CI, there will remain on the one hand □ KP on the other the ▭ DIE equal among themselves as above. Therefore the square of KI will be to the square of KP (*i. e.* to the ▭ DIE) as the square of BC to the square of B*n* (*i. e.* to the square of CE) as in the former case: And since here also the same may be demonstrated after the same manner of any other semiordinate K*i*; we may infer as above, that the □ □ KI and K*i* are to one another as the rectangles DIE and D*i*E, *&c.*

But after what way the same ellipses may be described by these right lined angles without any of these rulers thro' infinite points given, will be be manifest from the same figures to any attentive Person For having once determined the angle NCE or *n*CD (*num.* 2. *e. g.*) if NL or *nl* be applyed where you please by help of a pair of compasses, and continued to K,

K, ſo that LK or *lk* ſhall be equal to *l*B, you will have every where the point K, *&c.*

CONSECTARY IV.

SInce in the hyperbola (*Fig.* 123.) the $\square CN - \square CE = \square DNE$, and in the ellipſis (*Fig.* 124.) $\square CE - \square CN = \square DNE$, by vertue of *Prop.* 9 and 8. *lib.* 1 if for CN you put on both ſides for brevity's ſake *m*, then will the $\square$ DNE in the hyperbola be rightly expreſſed in theſe terms $\frac{mm - oobb}{4}$, and in the ellipſis in theſe $\frac{oobb - mm}{4}$.

Propoſition VI.

IN the parabola (α) *the Latus Rectum is to the ſum of two ſemiordinates* (*e. g.* IK + FG i. e. HO *in Fig.* 122.) *as their difference* (OG) *to the difference of the abſciſſa's* (IF *or* KO.)

(α) 3. *Property of the Parab.*

Demonſtration.

For if the greater abſciſſa EF be made $= ib$, and the leſs EI $= eb$, the ſemiordinates anſwering to them FG and IK will be $\sqrt{ooic}$ and $\sqrt{oecc}$ as is deduc'd from *Prop.* 1. Wherefore if you ſet in the ſame ſeries

1	2	3	4
The *Latus R.*	— ſum of the ſemiord.	— their diff.	— diff. of the abſciſſ.
$\frac{occ}{b}$	— $\sqrt{oicc} + \sqrt{oecc}$	— $\sqrt{oicc} - \sqrt{oecc}$	— $ib - eb$

and multiply the extremes and means, you'l have on both ſides the ſame product $oicc - oecc$, which will prove by vertue of *Prop.* 19. *lib.* 1. the proportionality of the ſaid quantities. Q. E. D.

SCHOLIUM.

THis is that property of the parabola, whereon the *Clavis Geometrica Catholica* of Mr. *Thomas Baker* is founded, which as unknown to the ancients, nor yet taken notice of by *Des Cartes*, he thinks was the reason why that incomparable Wit could not hit upon those universal rules for solving all Equations howsoever affected. Concerning which we shall speak further in its place. We will only further here note, that *Baker* was not the first Inventor of this property, but had it, as he himself ingeniously confesses, out of a Manuscript communicated to him by *Tho. Strode* of *Maperton*, Esquire.

Proposition VII.

IN the hyperbola and ellipsis (a) *the Latus Rectum is to the Latus Transversum, as the square of any semiordinate (e. g.* IK *in Fig.* 118 *and* 120.) *to the rectangle* (DIE) *contained under the lines intercepted between it and the Vertex's of the Latus Transversum.*

(a) *The* 3. *Property of the Hyperb. and Ellips. Apollon. lib.* 1. *Prop.* 21. *pars prior.*

Demonstration.

For the *Latus Rectum* is on both sides $\frac{ocd}{b}$, the *Latus Transversum ob*, &c. Wherefore if you make in the same series as the *Latus R.* to the *Lat. Transv.* so the □ IK to ▭ DIE

$\frac{ocd}{b}$ — *ob* —— in hyperb. *oecd* † *cecd* —— *oebb* † *eebb*

in ellips. *oecd* — *eecd* —— *oebb* — *eebb*

The rectangles of the extremes and means will both be *ooebcd* † *oeebcd*, and so will prove the proportionality of the said quantities, by *Prop.* 19. *lib.* 1. Q. E. D.

CON-

CONSECTARY I.

HEnce having given in the ellipsis (see *Fig.* 124) the *Latus Rectum* and the transverse ax, you may easily obtain the second ax or diameter, if you make

as the *Lat. Transv.* to the *Lat. Rect.* so the □ DCE to □ AC

$$ob \quad — \quad \frac{ocd}{b} \quad — \quad \frac{oobb}{4}\ F. \quad — \quad \frac{oocd.}{4}$$

CONSECTARY II.

THerefore the □ of the whole AB will be $= oocd =$ □ of the *Latus Rectum* into the *Lat. Transv.* (which *Apllonius* calls the *Figure*) so that the second Ax (and any second Diameter) will be a mean proportional between the *Latus Rectum* and the *Latus Transversum.* Hence in the hyperbola also the second or conjugate diameter may be called a mean proportional between the *Latus Rectum* and *Transversum*, i.e. $\sqrt{oocd}$ or a line which is equal in power to the *Figure*, as *Apollonius* speaks.

SCHOLIUM I.

HEnce may be derived another and more simple way of delineating organically the ellipsis in *Plano* about the given axes AB, DE (*Fig* 127.) which *Schooten* has given us; *viz.* by the help of two equal rulers CG and GK moveable about the points G and G: If, *viz.* the portions CF and HK are equal to half the lesser ax AC, but taken with both the augments (*viz.* CF + FG + GH) may $= \frac{1}{2}$ the greater ax CD or CI; and the point K moving along the produced line DE the point H may describe the curve EHAD. That this will be an ellipsis will be evident by vertue of this seventh *Prop.* from a property that agrees to this curve in all its points H. For having drawn circles about each diameter, and the lines IHN, FO perpendicular to CE; and having made the *Latus Rectum* EL, which is a third proportional to DE and AB by the second Consect. of this Prop. *&c.* by reason of the similitude of the triangles CFO, CIN, FO will be to FC as IN

IN to IC, and alternatively FO to IN as FC to IC *i. e.* as AC to CE or AB to DE. Therefore also the square of FO (or HN) will be to the square of IN, as the square of AB to the square of DE, by *Prop.* 22. *lib.* 1. *i. e.* as EL the *Latus Rectum* to ED the *Latus Transversum,* by *Prop.* 35. *lib.* 1. But the □ IN is = DNE, from the proporty of the circle. Therefore □ FO (or of the semiordinate HN) is to the ▭ DNE as EL the *Latus Rectum* to ED the *Latus Transv.* therefore by vertue of the pres. *Prop.* the point H is in the *Ellipsis,* and so any other, *&c.* Q. E. D.

CONSECTARY III.

NOW if in the ellipsis the □ of AC the second Ax ($= \frac{oocd}{4}$ by *Consect.* 1.) and □ CN the distance of the *Focus* from the centre ($= \frac{oobb - oocd}{4}$ by *Consect* 3. *Prop.* 5. the figure whereof you may see *n.* 124.) be joined in one sum; the □ AN will be $= \frac{oobb}{4}$, and so the line AN $= \frac{ob}{2}$ i. e. to half the *Latus Transversum:* So that hence having the axes given you may find the *Foci,* if from A at the interval CD you cut the transverse ax in N and N.

CONSECTARY IV.

NOW if, on the contrary, in an hyperbola (*Fig.* 123.) the □ AC or EF $= \frac{oocd}{4}$ be substracted from the □ CF or CN $= \frac{oobb + oocd}{4}$ by vertue of *Consect.* 2. *Prop.* 5. there will remain $\frac{oobb}{4}$ and its root $\frac{ob}{2}$, *i. e.* half the *Latus Transversum* CD: So that here also, the axes being given, you may find the *Focus*'s, if from the vertex E you make EF a perpendicular to the ax = to the second Ax AC, and at the interval CF from the centre C you cut the *Latus Transversum* continued in N and N.

SCHO-

SCHOLIUM II.

BUT now, that the right lines KN and KN drawn from any other point (*e. g.* K) to the *Foci*, when taken together in the ellipsis, but when ſubſtracted the one from the other in the hyperbola, are equal to the *Latus Tranſverſum* DE, we will a little after demonſtrate more univerſally, and alſo ſhew an eaſie and plain *Praxis* of delineating the ellipſis and hyperbola in *Plano*, having the axes and conſequently the *Foci* given.

CONSECTARY V.

SInce we have before demonſtrated *Conſect.* 2. and 3. *Prop.* 5. that the □ DNE in the hyperbola and alſo in the ellipſis is $= \frac{oocd}{4}$; and here in *Conſect.* 1. the □ of the ſecond ſemi-diameter AC is alſo $= \frac{oocd}{4}$; it is evident that this □ AC is equal to the □ DNE.

CONSECTARY VI.

IT is hence moreover evident, if the ſquare of half the tranſverſe diameter GE $= \frac{oobb}{4}$ be compared with the ſquare of half the ſecond diameter AC or EF $= \frac{oocd}{4}$, multiplying both ſides by 4 and dividing by o; they will be to one another as obb to ocd i. e. further dividing both ſides by b, as ob to $\frac{ocd}{b}$ the *Latus Tranſverſum* to the *Latus Rectum*.

CONSECTARY VII.

BUT ſince alſo the □ DIE is to the □ IK as the *Latus Tranſverſum* to the *Latus Rectum*, by vertue of the preſent 7. *Prop.* the ſquare of CE the tranſverſe ſemidiam. will be to

to the ſquare of AC the ſecond ſemidiam. (or by the 5*th. Conſect.* of this, to the □ DNE) as the □ DIE to the ſquare of IK.

CONSECTARY VIII.

YOU may alſo now have the □ IK (which otherwiſe in the hyperb. is *eocd* † *eecd*, in the ellipſe *oecd*—*eecd*, by vertue of *Prop.* 2. and 3.) in other terms, if you make as □ CE to the □ DNE ſo the □ DIE to a fourth; *i. e.*

as $\frac{oobb}{4}$ to $mm - \frac{oobb}{4}$ } by vertue of *Conſect.* 4. *Prop.* 5.
(ſo *oebb* † *eebb* in the hyperb.
as $\frac{oobb}{4}$ to $\frac{oobb}{4} - mm$ ſo *oebb*—*eebb* in the ellipſis.

For hence by the Golden Rule the ſquare IK may be infer'd as a fourth proportional.

In the hyperbola $\frac{4emm}{o} \dagger \frac{4eemm}{oo} - oebb - eebb$;

In the ellipſis $oebb - eebb - \frac{4emm}{o} \dagger \frac{4eemm}{oo}$:

The uſe of which quantities will preſently appear.

Propoſition VIII.

(a) *Apollon. Lib.* 3. *Prop.* 51. *and* 52.

THE { *Aggregate in the ellipſe* / *Difference in the hyperb.* } *of the right lines* (a) *KN and Kn* (*Fig.* 128.) *drawn from the ſame point K to both the Focus's is equal to the tranſverſe ax DE.*

An Ocular Demonſtration.

WHich conſiſts wholly in this to find the lines KN and K*n* by help of the right-angled triangles IKN and IK*n* (ſc. the hypothenuſes having the ſides given) and afterwards ſee if the ſum of both in the ellipſe, and difference in the hyperbola be = *ob*, *i. e.* to the tranſverſe ax DE.

I. *In the Ellipsis.*

Putting for CN (which above *Prop.* 5. *Cons.* 3. was found to be $\sqrt{\frac{oobb - oocd}{4}}$) I say putting for it *m*, you'l have

$IN = CI + CN = \frac{1}{2}ob - eb + m$
$In = Cn - CI = m - \frac{1}{2}ob + eb$
Therefore $\square IN = \frac{1}{4}oobb - oebb + eebb + obm - 2ebm + mm$
$\square In = \frac{1}{4}oobb - oebb + eebb - obm + 2ebm + mm$
Add to each $\square IK$ which was found in *Prop.* 7. *Consect.* 8. in the ellipsis $= oebb - eebb - \frac{4emm}{o} + \frac{4eemm}{oo}$ and you'l have

$$\square KN = \frac{1}{4}oobb + obm - 2ebm + mm - \frac{4emm}{o} + \frac{4eemm}{oo}$$

and by extracting the roots (which is easie) you'l have
$KN = \frac{1}{2}ob + m - \frac{2em}{o}$ and

$Kn = \frac{1}{2}ob - m + \frac{2em}{o}$;

Sum *ob*. Q. E. D.

II. *In the Hyperbola.*

Putting again *m* for CN (which above *Cons.* 2. *Prop.* 5. was found to be $\sqrt{\frac{oodc + oobb}{4}}$) and you'l have

$IN = CI + CN = \frac{1}{2}ob + eb + m$
$In = CI - Cn = \frac{1}{2}ob + eb - m$. Therefore
$\square IN = \frac{1}{4}oobb + oebb + eebb + obm + 2ebm + mm$ and
$\square In = \frac{1}{4}oobb + oebb + eebb - obm - 2ebm + mm$
Add to both the $\square IK$ which was found in *Prop.* 7. *Consect.* 8. in the hyperbola $\frac{4emm}{o} + \frac{4eemm}{oo} - oebb - eebb$ and you'l have
$$\square KN = \frac{1}{4}oobb + obm + 2ebm + mm + \frac{4emm}{o} + \frac{4eemm}{oo}$$

Q $\square$ Kn

$\square Kn = \frac{1}{4} oobb - obm - 2ebm + mm + \frac{4emm}{o} + \frac{4eemm}{oo}$

and extracting the roots out of these (which is easie) you'l have $KN = \frac{1}{2} ob + m + \frac{2em}{o}$

$$Kn = \frac{1}{2} ob - m - \frac{2em}{o}$$

(which is a false or impossible root, for it would be CE—CN and moreover —— another quantity.

Or $Kn = m + \frac{2em}{o} - \frac{1}{2} ob$; which is a true and possible root.

The difference therefore of the true roots is $= ob$. Q. E. D.

SCHOLIUM I.

WE first of all tried to make a literal Demonstration by using the quantity of the square IK as you have it expressed *Prop.* 2. and 3. and the quantity IN as it was compounded of $CI = ob + eb + CN = \sqrt{\frac{oocd + oobb}{4}}$, *&c.* but we found it very tedious in making only the squares of IN and I*n*. Then for the surd quantity CN we substituted another, *viz.* *m*, and we produced the squares of IN and I*n* as above, but we added the square of IK in its first value; and thus we obtain'd the squares KN and K*n*, but in such terms, that the exact roots could not be extracted, but must be exhibited as surd quantities, and consequently we must make use of the rules belonging to them to find their sum or difference, which we laid down *Cons.* 3. *Prop.* 7. and *Consect.* *Prop.* 10. *Lib.* 1. which tho' it would succeed, yet wou'd be full of trouble and tediousness. Therefore at length when we came to use those other terms which express the square IK, the business succeeded as easie as we could wish, and that in a plain and easie way and no less pleasant, which I doubt not but will also be the opinion of the Reader, who shall compare this with other demonstrations of the same thing, which only lead indirectly to this truth, or with them which *de Witte* has given us in *Elem. Curvar. lin. p. m.* 293. and 302. and which he thinks easie

eaſie and ſhort enough in reſpect of others both of the ancients and moderns, and which we have reduced into this yet more diſtinct form, and accommodated to our ſchemes.

Preparation for the Hyperbola.

Make as $\left\{\begin{matrix} \text{CD to CN} \\ \text{EE — CN} \end{matrix}\right\}$ ſo CI to CM

ſo that the $\Box$ of $\left\{\begin{matrix} \text{DCM} \\ \text{MCE} \end{matrix}\right\}$ will be $=$ $\Box$ $\left\{\begin{matrix} \text{NCI} \\ n\text{CI.} \end{matrix}\right.$

Becauſe therefore it will be by *Conſect.* 7. *Prop.* 7.
as $\Box$ CD to $\Box$ DNE, ſo the $\Box$ DIE to the $\Box$ IK.

And alſo by compoſition,

as $\Box$ CD to $\left\{\begin{matrix} \Box\ \text{CD} + \Box\ \text{DNE} \\ \textit{i. e.}\ \Box\ \text{CN } \textit{per } 9.\ \textit{lib } 1. \end{matrix}\right\}$ ſo DIE to DIE $+$ $\Box$ IK.

Therefore by a Syllepſis,

as $\Box$CD to the $\Box$CN ſo $\left\{\begin{matrix} \Box\ \text{CD} + \text{DIE} \\ \textit{i. e.}\ \Box\ \text{CI} \end{matrix}\right\}$ to $\Box$ CN $+$ DIE $+$ $\Box$ IK.

But alſo by the Hypotheſis.
as the $\Box$ CD to the $\Box$ CN ſo $\Box$ CI to $\Box$ CM. Therefore
$\Box$ CM is $=$ $\Box$ CN $+$ DIE $+$ $\Box$ IK.

Demonſtration.

Since therefore it is certain that the difference between DM and EM is the tranſverſe ax DE; if it be demonſtrated that DM is $=$ KN and EM $=$ Kn, the buſineſs will be done, becauſe the difference between KN and Kn is alſo the tranſverſe ax DE.

Reſolve the $\Box$ *KN.*

It is certain that NIq $+$ IKq $=$ KNq.
Subſtitute for NIq, by the 7. *Lib.* 1. CIq $+$ CNq $+$ 2NCI.

Preparation for the Ellipſis.

Make as CD to CN ſo CI to CM.
So that the $\Box$ DCM is $=$ $\Box$ NCI.
Becauſe therefore by *Conſect.* 7. *Prop.* 7.
as $\Box$ CD to $\Box$ DNE ſo $\Box$ DIE to the $\Box$ IK;

 Then

Then also by dividing,

as $\square$ CD to $\left\{\begin{array}{l}\square\ CD - DNE \\ i.e.\ \square\ CN,\ by\ 8.\ l.\ 1.\end{array}\right\}$ so DIE to DIE — $\square$ IK.

Therefore by a Dialepsis,

as $\square$ CD to $\square$ CN, so $\left\{\begin{array}{l}\square\ CD - \square\ DIE \\ i.e.\ CI\ \square\ by\ 8.\ cit.\end{array}\right\}$ to $\square$ CN — DIE + $\square$ IK.

But also by the Hypothesis,

as $\square$ CD to $\square$ CN, so $\square$ CI to $\square$ CM :

Therefore $\square$ CM is = $\square$ CN — DIE + $\square$ IK.

Demonstration.

Since therefore it is certain that the sum of DM and EM is the transverse ax DE; if it be demonstrated that DM is = KN and EM = K*n*, the business will be done, because the sum of KN and K*n* is also equal to the transverse ax DE.

Resolve the $\square$ *KN.*

It is certain that NI*q* + IK*q* = KN*q*.

Substitute for NI*q*, by the 7. *lib.* 1. CI*q* + CN*q* + 2NCI.

Then will CI*q* + CN*q* + 2NCI + IK*q* = KN*q*.

Substitute for CI*q*, by the 9. *lib.* 1. CD*q* + DIE; then will CD*q* + DIE + CN*q* + 2NCI + IK*q* = KN*q*.

Resolve also $\square$ *DM.*

It is certain that CM*q* + CD*q* + $\left\{\begin{array}{l}2DCM \\ 2NCI\end{array}\right\}$ = DM, by the 7 *lib.* 1.

Substitute for CM*q* its value by the Preparation, and you'l have

CN*q* — DIE + IK*q* + CD*q* + 2NCI = DM*q*:

Which were before = KN*q*.

Therefore KN = DM; which is one.

In like manner resolve $\square$ *Kn.*

It is certain that *n*I*q* + IK*q* = K*nq*.

Substitute for *n*I*q*, by *Consect.* 1. *Prop.* 10. *Lib.* 1. CI*q* + CN*q* — 2*n*CI, and you'l have

CI*q* + CN*q* — 2*n*CI + IK*q* = K*nq*.

Substitute for CI*q*, by the 9. *lib.* 1. CD*q* + DIE, and you'l have CD*q* + DIE + CN*q* — 2*n*CI + IK*q* = K*nq*.

Resolve

Resolve also the □ *EM.*

It is certain that $2CDq + 2CMq - DMq = EMq$ *per* 13. *lib.* 1.

Then will $CIq + CNq + 2NCI + IKq = KNq$.

Substitute for CIq by the 8. *lib.* 1. $CDq - DIE$; then will $CDq - DIE + CNq + 2NCI + IKq = KNq$.

Resolve also the □ *DM.*

It is certain that $CMq + CDq + \left\{\begin{matrix} 2DCM \\ 2NCI \end{matrix}\right\} = DMq$ *per* 7. *lib.* 1.

Substitute for CMq its value from the preparation, and you'l have

$CNq - DIE + IKq + CDq + 2NCI = DMq$:

Which before were $= KNq$.

Therefore $KN = DM$; which is one.

In like manner resolve the □ *Kn.*

It is certain that $nIq + IKq = Knq$.

Substitute for nIq by *Consect.* 1. *Prop.* 10. *lib.* 1.

$CIq + CNq - 2NCI$, and you'l have

$CIq + CNq - 2NCI + IKq = Knq$.

Substitute for CIq *per* 8. *lib.* 1. $CDq - DIE$, and you'l have

$CDq - DIE + CNq - 2NCI + IKq = Knq$.

Resolve also □ *EM.*

It is certain that $2CDq + 2CMq - DMq = EMq$ *per* 13. 1.

Substitute the value of DMq first found above, and you'l have

$CDq + CMq - 2nCI = EMq$.

Substitute for CMq the value as in the preparation, and you'l have

$CDq + CNq + DIE - 2nCI + IKq = EMq$:

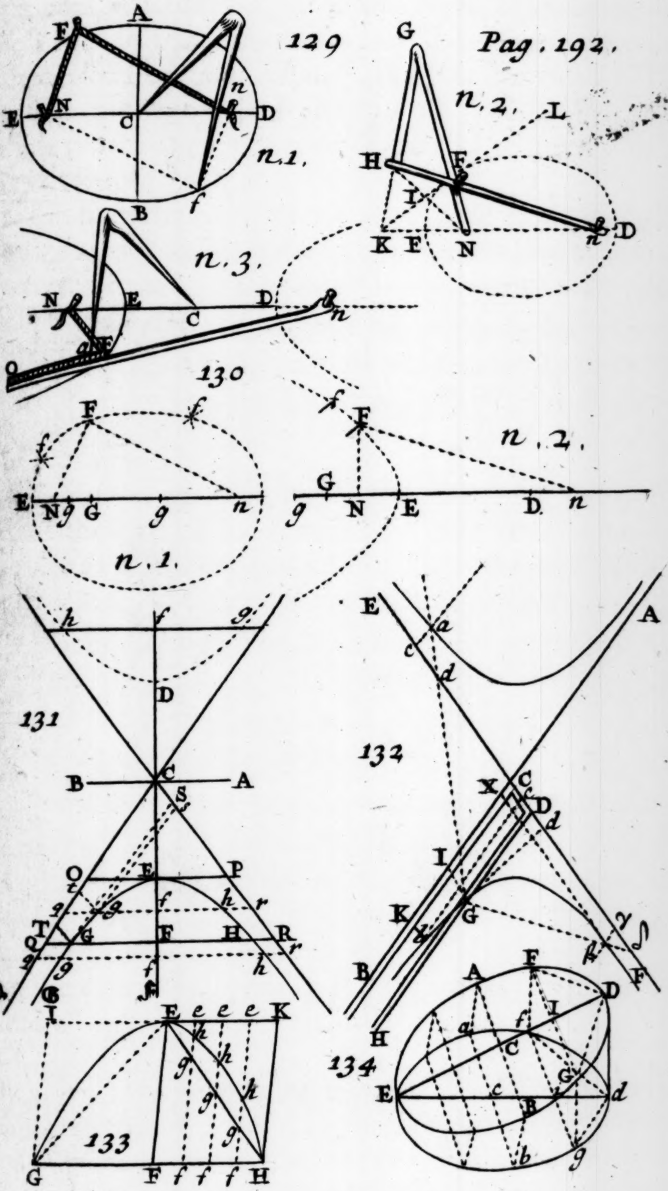
Which were before $= Knq$.

Therefore $Kn = EM$; which is the other.

SCHO-

SHCOLIVM II.

HEnce you have the common mechanical ways of describing the ellipsis and hyperbola about their given axes; and the ellipsis, if the *Foci* N, N, (*Fig*. 129. *n*. 1.) are given, or found according to *Consect*. 3. *Prop*. 7. and having therein stuck or fixed two pins, put over them a thread NF*n* tyed both ends together precisely of the length you design the greater ax DE to be of, and having put your pencil or pen in that △-string draw it round, always keeping it equally extended or tight. Now because the parts or portions of the thread remain always equal to the whole ax DE, what we proposed is evident by the present *Prop*. which may also be very elegantly described by a certain sort of Compasses, a description whereof *Swenterus* gives us in his *Delic. Physico-Math. Part*. 2. *Prop*. 20. which may be also done by a sort of organical Mechanism, by the help of two rulers moveable in the *Foci* GN and H*n* (*n* 2.) and equal to the transverse ax DE, and fastned above by a transverse ruler GH equal to the distance of the *Foci*, as may appear from the Figure. For if the style F be moved round within the fissures of the cross rulers H*n* and GN the curve thereby described will be an ellipsis from the property we have just now demonstrated of it, which it hath in every point F. For the triangles HGN and NH*n*, which have one common side HN, and the others equal by construction, are equal one to another, and consequently the angles FHN and FNH equal, so also the legs HF and FN, and so likewise FN and F*n* together are equal to H*n* = DE; which is the very property of the ellipse we are now treating of. But *Van Schooten*, who taught us this delineation, hints, that, if thro' the middle point I of the line HN you draw the line IFL, it will touch the ellipsis in the point F; for since the angles IFH and IFN are equal, by what we have just now said, the vertical angle LF*n* of the one IFH, will be necessarily equal to the other IFN: But this equality of the angles, made by the line KL drawn thro' F, with both those drawn from the centres, is here a sign of contact, as is in the circle the equality of the angles with a line drawn from its one centre. So that after this way you may draw a tangent thro' any given point F of the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the ellipſis without this organical *apparatus* of Rulers; *viz.* if, having drawn from both the *Focus's* thro' the given point F the right lines *n*H, NG equal to the *Latus Tranſverſum* DE, you biſect HN in I and draw IFL: Or if the line that connects the extremes GH be produced to K, and you draw thence KFL, *viz.* in that caſe where GH and N*n* are not parallel; otherwiſe a line drawn thro' the point F parallel to them would be the tangent ſought.

As to the hyperbola, there is a mechanick method of drawing that alſo, not unlike the others, from a like property in that, communicated by the ſame *Van Schooten*, *viz.* If having found the *Focus's* N and *n* (*Fig.* 129. *n.* 3.) you tye a thread NFO in the *Focus* N and at the end of the ruler *n*O of the length of the tranſverſe ax DE; then putting in a pen or the moveable leg of a pair of compaſſes (nor would it be difficult to accommodate the practice we before made uſe of to this alſo) draw or move it within the thread NFO from O to E, ſo that the part of the thread NO may always keep cloſe to the ruler as if it were glued to it. For if we call the length of the thread X, and the tranſverſe ax *ob* as above, the ruler *n*O will be, by the Hypoth. $= X + ob$. Make now the part of the thread OF $= \frac{1}{2} X$, the remainder or other part will be NF $= \frac{1}{2} X$ and *n*F $= \frac{1}{2} X + ob$, and the difference between FN and F*n*, $= ob$. Make OF $= \frac{3}{4} X$, then will FN be $\frac{1}{2} X$ and F*n* $\frac{1}{4} X + ob$, the difference ſtill remaining *ob* and ſo *ad infinitum.* In ſhort, ſince the difference of the whole thread and of the whole ruler is *ob*, and in drawing them, the ſame OF is taken from both, there will always be the ſame difference of the remainders. Hence alſo aſſuming at pleaſure the points N and *n* you may deſcribe hyperbola's ſo, the thread NFO be ſhorter than the ruler *n*FO: For if it were equal there would be deſcribed a right line perpendicular to N*n*, thro' the middle point C.

There yet remains one method of deſcribing hyperbola's and ellipſes in *Plano*, by finding the ſeveral points without the help or *Apparatus* of any threads or inſtruments, *viz.* in the ellipſis, having given or aſſumed the tranſverſe axis DE and the *Foci* N and *n* (*Fig.* 130. *n.* 1.) if from N at any arbitrary diſtance, but not greater than half the tranſverſe ax NF, you make an arch, and keeping the ſame opening of the compaſſes

passes you cut off, from the transverse ax, EG, and then, taking the remaining interval GD, from *n* you make another arch cutting the former in F, and so you will have one point of the ellipse, and after the same way you may have innumerable others, *f*, *f*, *f*, *&c.*

In like manner to delineate the hyperbola, having given or assumed the transverse ax DE and the *Focus's* N and *n* (*n.* 2.) if from N at any arbitrary distance NF you strike an arch, and keeping the same aperture of the compasses from the diameter continued, you cut off EG, and then at the interval GD from *n* make another arch cutting the former in F, you will have one point of the hyperbola, and after the same way innumerable others, *f*, *f*, *&c.*

Proposition IX.

If the secondary ax, or conjugate diameter of the hyperbola AB (Fig. 131.*) be applied parallel to the vertex* E, *so that it may touch the hyperbola, and* OE, EP *are equal like* BC *and* AC, *and from the centre* C *you draw thro'* O *and* P *right lines running on* ad infinitum, *and lastly* QR *parallel to the Tangent* OP; *you'l have the following*

CONSECTARYS.

I. The parts QG and HR intercepted between the curve and those right lines CQ, CR will be equal; for by reason of the similitude of the △ △ CEP and CFR as also CEO, CFQ as CE is to EO (and EP) so will CF be to FQ and FR, and consequently these will be equal; and so taking away the semiordinates FG and FH which are also equal, the remainders GQ and HR will be also equal, and consequently the □ □ QGR, GRH, *&c.* all equal among themselves: Which we had already deduced before in *Consect.* 2. and 3. *Def.* 7.

II. The rectangle QGR will be $=$ □ EO or EP $= \frac{ocd}{4}$ i.e. (as *Apollonius* speaks) to the fourth part of the figure: For by reason

reason of the similitude of the △ △ CEO, CFQ, CE will be to EO as CF to FQ: *i. e.*

as the □ CE to the □ EO
i.e. as (by *Conf* 2. 7.) the *Lat. Transv.* to the *Lat. Rect.*
i.e. (by the 7. *Prop.*) as the ▭ DFE to the □ FG
} so the □ CF to the □ FQ

But now if from the □ CF you take the ▭ DFE, there will remain the □ CE, by *Prop.* 9. *lib.* 1. and if from the □ FQ you take the ▭ FG there will remain the ▭ QGR, by *Prop.* 8. *lib.* 1. wherefore that remaining □ CE to this remaining ▭ QGR will be, as was the whole square CF to the whole square FQ, by *Prop.* 26. *lib.* 1. *i. e.* as was the □ CE to the □ EO; consequently the ▭ QGR and the square EO (to which the same square CE bears the same proportion) will be equal among themselves.

III. Since this is also after the same manner certain of any other rectangle *ggr* or *grh*, *&c.* it follows that all such rectangles are equal among themselves.

IV. Wherefore it is most evident, since the lines FR, *fr*, *&c.* and so GR and *gr* grow so much the longer, by how much the more remote they are from the vertex E; that on the contrary the lines QG and *qg* must necessarily so much the more decrease and grow shorter, and consequently the right line CQ approach so much nearer and nearer to the curve EG.

V. But that they can never meet or coincide altho' produced *ad infinitum* will thus appear; if it were possible there could be any concourse or meeting, so that the point G and Q or *g* and *q* could any where coincide, it would follow from *Consect.* 2. that as the ▭ DFE to the square FG so the square CF to the square FQ *i e.* to the same square FG; and so that the ▭ DFE would be = □ CF; which is absurd by *Prop.* 9 *lib.* 1. so that now it is evident that the lines COQ and CPR drawn according to *Consect.* 1. are really Asymptotes (*i. e.* they will never (*a*) coincide (*viz.* with the curve of the hyperbola) as *Apollonius* has named them.

(*a*) *Apoll. Prop.* 1. *lib.* 2.

R VI. Having

VI. Having drawn the right lines from G and *g* parallel to both the asymptotes, *viz.* GS and *gs* and likewise GT and *gt*, the rectangles TGS and *tgs* will be (α) equal among themselves. For by reason of the similitude of the △ △ TQG and *tqg*, *first*, TG will be to QG as *tg* to *qg*; and, by reason of the equality of the □ □ QGR and *qgr*, *secondly*, QG will be to *gr* reciprocally as *qg* to GR, by *Prop* 19. *lib.* 1. and by reason of the similitude of the △ △ SGR and *sgr*, *thirdly*, as *gr* to *gs* so GR to GS, wherefore (since in two series

(α) *Apollon. Prop.*12. *lib.*2.

1. 2. 3.
as TG to QG to *gr* to *gs*
so *tg* to *qg* to GR to GS)

you'l have *ex æquo* or by proportion of equality as TG to *gs* so *tg* to GS, by *Prop.* 24 *lib.* 1. Therefore, by *Prop.* 17. of the same, the □ of TG into GS = □ of *tg* into *gs*. Q. E. D.

SCHOLIUM.

HEnce, lastly, we have a new genesis of the hyperbola in *Plano* about its given diameters from the speculations of (β) *De Witt*; if, *viz.* having drawn the lines AB and EF cross one another at pleasure (*Fig.* 132) to the angle BCF you conform the moveable angle BCD (*acd* being to be delineated in the opposite hyperbola equal to the contiguous ACD) one of whose legs is conceived to be indefinitely extended, but the other CD of any arbitrary length; and to the end of it D apply the slit of a moveable ruler GD about the point G at any arbitrary interval GD (but yet parallel to the leg CB in this first station) and so carrying together along with it the moveable angle BCD about the line ECF, but so that the leg CD may always remain fast to it, and the other CB be intersected in its progress by the ruler GDH, *e. g.* in *b* or β This point of intersection, thus continually moved on, will describe the curve *b*Gβ, which we thus prove to be an hyperbola: Because the ruler GDH turning about the pole G, and carried from D *e. g.* to *d* or δ cuts the leg of the moveable angle CB brought

(β) *De Witt Elem. Curv. lib.* 1. *cap.* 2. *prop.* 3.

brought to the ſituation cb or $\gamma\beta$, and in the mean while remaining always parallel to it ſelf; and having drawn from the points of interſection b and β and G the lines GI, bK and $\beta\kappa$ parallel to the ruler CF, becauſe *e. g.* in the ſecond ſtation, having taken the common quantity cD from the equal ones CD and cd, the remainders Cc and Dd are equal, and by reaſon of the ſimilitude of the $\triangle\ \triangle$ dcb and dDG,

$$\left.\begin{array}{l}\text{as } \mathrm{D}d\\ i.e\ \mathrm{C}c\\ \text{or } b\mathrm{K}\end{array}\right\}\text{ to DG, ſo } \left.\begin{array}{l}dc\\ i.e.\ \mathrm{DC}\\ \text{or GI}\end{array}\right\}\text{ to } cb;$$

the rectangle of Kb into bc will be $=$ $\square$ of DG into GI, by *Prop.* 18. *lib.* 1. and in like manner, when in the third ſtation having added the common line Dγ to the equal ones CD and $\gamma\delta$, the whole lines Dδ and Cγ are equal, and, by reaſon of the ſimilitude of the $\triangle\ \triangle$ $\beta\gamma\delta$ and GDδ

$$\left.\begin{array}{l}\text{as } \mathrm{D}\delta\\ i.e.\ \mathrm{C}\gamma\\ \text{or } \beta\kappa\end{array}\right\}\text{ is to DG ſo is } \left.\begin{array}{l}\gamma\delta\\ i.\ e.\ \mathrm{DC}\\ \text{or GI}\end{array}\right\}\text{ to } \gamma\beta;$$

the $\square$ of $\kappa\beta$ into $\beta\gamma$ $=$ $\square$ of DG into GI, by the ſame 18. *Prop.* Wherefore the three points b, G, β, (and ſo all the others that may be determined the ſame way) are in the hyperbola, whoſe aſymptotes are CB and CF and its centre C, *&c.* by the preſent *Prop. Conſect.* 6. Q. E. D.

You may alſo determine innumerable points of this curve ſeparately without the motion we have now preſcrib'd, *viz.* as the point a in the oppoſite hyperbola, if thro' any aſſumed point c in the aſymptote CE you draw a parallel to the other aſymptote CA, and having made cd equal to CD, from G thro' d draw Gda, and ſo in others.

R 2 CHAP.

CHAP. II.

Of Parabolical, Hyperbolical and Elliptical Spaces.

Proposition X.

(a) *Archim. de Quadratur. Parab. Prop.* 17. *and* 24.

THE (a) *Parabolick Space* (i. e. *in Fig.* 133: *that comprehended under the right line* GH *and the parabola* GEH) *is to a circumscribing Parallelogram* GK, *as* 4 *to* 6 (*or* 2 *to* 3) *but to an inscribed* △ GEH *as* 4 *to* 3.

Demonstration.

Suppose FH divided first into two then into four equal parts, and draw parallel to the ax EF the lines *ef*, *ef*, *&c.* dividing also EF into four parts, the first *fg* will be 3, the second 2, the third 1, by *Prop.* 34. *lib.* 1. but as *ef* is to *ge* so is *ge* to *he*, by *Consect.* 1. *Prop.* 4. Therefore *he* in the diameter EF is = *o*, in the first *ef* it is = $\frac{1}{4}$ (for as *ef*, 4, to *ge*, 1, so *ge*, 1, to *he*, $\frac{1}{4}$) in the second *ef* a portion of *he* is = $\frac{4}{4}$, in the third to $\frac{9}{4}$, and so the portions *eh* in the trilinear figure E*h*HK make a series in a duplicate arithmetical progression, *viz.* 1, 4, 9, 16: After the same manner, if the parts F*f*, *&c.* are bisected, you'l find the portions *eh* in the external trilinear figure to make this series of numbers $\frac{1}{8}$, $\frac{4}{8}$, $\frac{9}{8}$, $\frac{16}{8}$, $\frac{25}{8}$, $\frac{36}{8}$, $\frac{49}{8}$, $\frac{64}{8}$, and so onwards. Wherefore since the portions *eh* or the indivisibles of the trilinear space circumscribed about the parabola are always in a duplicate arithmetical progression: the sum of them all will be to the sum of as many indivisibles of the parallelogram FK, equal to the line KH, *i. e.* the trilinear space it self to this parallelogram as 1 to 3, by *Consect.* 10. *Prop.* 21. *lib.* 1. Wherefore the semi-parabola FE*h*H will be as 2, and the △ FEH as 1 $\frac{1}{2}$; therefore the whole parabola as 4, and the whole △ GEH as 3, and the whole parallelogram GK as 6. Q. E. D.

CON-

CONSECTARY I.

IT is evident (*a*) that in the first division, the second line *fh* (*i. e.* that drawn from the middle of the base FH) is three such parts whereof FE is 4; for *eh* is $\frac{4}{4}$ *i. e.* 1, therefore *fh* is 3.

(*a*) *Archim. Prop.* 19. *with the Coroll.*

CONSECTARY II.

IT is also evident, that this demonstration will hold of any parabolick segment.

Proposition XI.

THE Elliptical Space (a) comprehended by the Ellipsis DAEB (Fig. 127.) is to a circle described on the transverse ax DE, as the Axis Rectus or conjugate diameter AB to the transverse ax DE.

(*a*) *Archim. lib. de Conoid.* &c. *Prop.* 5.

Demonstration.

THis is in the first place evident from the genesis of the ellipse we deduced in *Schol.* 1. *Prop* 7. for in that deduction we shewed that FO, *i. e.* HN was to NI as AB to DE: Which since it is true of all the other indivisibles or ordinates HN and IN *ad infinitum*; it is manifest that the planes themselves constituted of these indivisibles will have the same reason among themselves, as the *Axis Rectus* AB to the transverse DE. Q. E. D.

CONSECTARY I.

THerefore the quadrature of the ellipse will be evident, if that of the circle be demonstrated.

CON-

CONSECTARY II.

SInce a circle described on the least diameter AB will be to one described on the greater diameter DE, as AB to a third proportional by *Prop.* 35. *lib.* 1. it follows by vertue of the present *Prop.* that the ellipse is a mean proportional between the greater and lesser circle, *i e.* as the ellipse is to the greater circle so is the lesser circle to it, *viz.* as AB to DE.

CONSECTARY III.

HEnce you may have a double method of determining the area of an ellipse. 1. If having found the area of the greater circle, you should infer, as the greater diameter of the ellipsis to the less, so the area of the circle found to the area of the ellipse sought. 2. If having also found the area of the lesser circle, you find a mean proportional between that and the area of the greater.

SCHOLIUM.

WE may also shew the last part of the second *Consect.* thus, 1. If having described the circle E*adb*E (*Fig.* 134.) about the least axis of the ellipse we conceive a regular hexagon to be inscribed, and an ellipse coinciding with one end E of its transverse ax, and with the other or opposite one D to be so elevated, that with the point *d* it may perpendicularly hang over the circle, and further from all the angles of the figure inscribed in the circle you erect, the perpendiculars *g*G, *b*B, *&c.* it is certain that the sides ED and E*d* of the triangle DE*d* will be cut by the parallel planes FG*gf*, *&c.* into proportional parts, and that those by reason of the similitude of the △ △ FDG and *fdg*, and so also the other rectangles will be among themselves as the intercepted parts of the lines ID and *id*, CI and *ci*, and *in infinitum*, (*viz.* of how many sides soever the inscribed figure consists:) Wherefore also all the parts of the ellipse taken together will be to all the parts of the circle taken together, *i. e.* the whole ellipse to the whole circle

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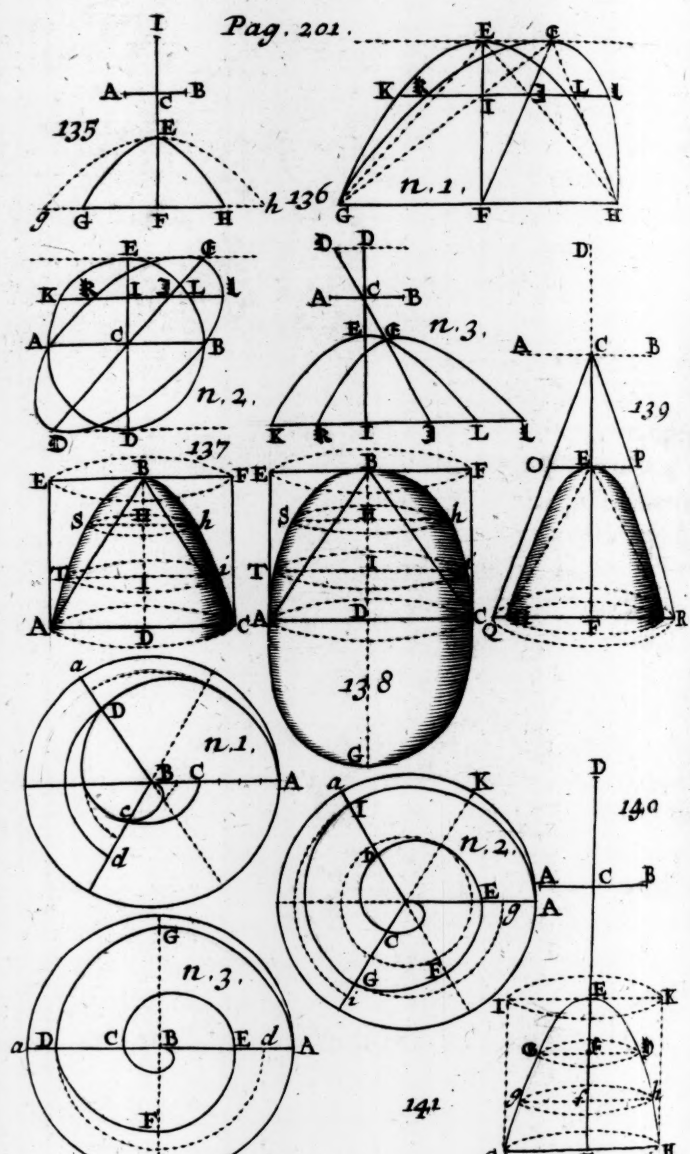

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circle as all the parts of the diameter ED or *ab*, *i. e.* as DE it ſelf to AB. Q. E D.

CONSECTARY IV.

IT is alſo evident that both theſe demonſtrations of the preſent *Prop.* will be alſo the ſame in any ſegments of the ellipſis or circle.

Propoſition XII.

ANY Hyperbolical ſpace GEHG *(Fig. 135) is to any Hyperbolick figure of equal heighth gEhg [whoſe Latus Rectum and Tranſverſum are equal (as in the circle) and alſo equal to the Latus Tranſverſum of the former* DE, *as the Axis Rectus (or conjugate)* AB *is to the Latus Tranſverſum* DE *(as in the ellipſis.)*

Demonſtration.

By the Hypoth. and *Prop.* 7. and its ſecond *Conſect.* the □ Fg is ∺ □ DFE. Wherefore this □ DFE *i e.* the □ Fg is to the □ FG as the *Latus Tranſverſum* to the *Latus Rectum* of the hyperbola GEHG, by the ſame ſeventh *Prop. i. e.* (by *Conſect.* 2 of the ſame) as the ſquare of the *Latus Tranſverſ.* DE to the ſquare of the conjugate AB : Therefore the roots of theſe ſquares will be alſo proportional, *viz.* Fg to FG as DE to Ab ; and conſequently (ſince the ſame is true of any other ordinates *ad infinitum*) the whole hyperbola *gEhg* will be to the whole one GEHG as DE to AB. Q. E. D.

CONSECTARY I.

THerefore having found the quadrature of ſuch an hyperbola, whoſe *Latus Rectum* and *Tranſverſum* are equal, you may have alſo the quadrature of any other hyperbola.

CON-

CONSECTARY II.

IT is evident that the same demonstration will hold in any other hyperbola's.

Proposition XIII.

ANY Parabolick segments upon the same base, and hyperbolical and elliptical ones described about the same conjugate (one whereof shall be a right *one, the other a* scalene) *and constitued between the same parallels, are equal.*

Demonstration.

I. It is evident of Parabola's; for both the right one GEHG, and the scalene one G$\mathfrak{E}$HG (*Fig.* 136. *n.* 1.) (for the demonstration of *Prop.* 10. will hold in both) is to a △ inscribed in them as 4 to 3. But the triangles GEH and G$\mathfrak{E}$H are equal, by *Consect.* 5. *Def.* 12. or *Prop.* 28. *lib.* 1. Therefore the Parabola's also.

Or thus, in the right parabola GEHG every thing is the same as in 1. and 4. *Prop* of this Book, *viz.* EI = *eb*, EF = *ib*, the square IK = *oecc*, the □ FG = *oicc*. And because therefore in the scalene Parabola also the square FG remains = *oicc*, make F$\mathfrak{E}$ = *n*, and find both the abscissa $\mathfrak{E}\mathfrak{I}$, and the □ answering to it $\mathfrak{I}\mathfrak{K}$.

1. For the abscissa; as FE to EI so F$\mathfrak{E}$ to $\mathfrak{E}\mathfrak{I}$, *per*

$$ib \text{ —— } eb \text{ —— } n \text{ —— } \frac{en}{i}$$

Consect. 4 *Prop.* 34 *lib.* 1.

2. For the □ $\mathfrak{I}\mathfrak{K}$; as F$\mathfrak{E}$ to $\mathfrak{E}\mathfrak{I}$ so □ FG to □ $\mathfrak{I}\mathfrak{K}$,

$$n \text{ —— } \frac{en}{i} \text{ —— } oicc \text{ —— F. } oecc.$$

per Prop. 4. of this.

Therefore the □ $\mathfrak{I}\mathfrak{K}$ = □ IK and $\mathfrak{I}\mathfrak{K}$ = IK, and this in any case *ad infinitum*: Therefore the one parabola is = to the other. Q. E. D.

II. The

II. The buſineſs is much after the ſame way evident of ellipſes and hyperbolas. For making all things in the ellipſis and right hyperbola (*n.* 2. and 3. *Fig.* 136.) as in *Prop.* 2, 3, 5, 7. *viz.* the □ IK *oecd* —— *eecd* in the ellipſis, *oecd* † *eecd* in the hyperbola, the □ AB *oocd* by *Conſect.* 2. *Prop.* 7. EI = *eb*, DE = *ob*, *&c.* if in oblique ones for the *Latus Tranſverſum* $\mathfrak{DE}$ you put *n*, and ſeek the *Latus Rectum* and abſciſſa $\mathfrak{EI}$, you may by means of theſe alſo have the ſquare $\mathfrak{IK}$, by *Prop.* 2. and 3.

1. *For the Latus Rectum.*

As *n* to ϒ $\overline{oocd}$ ſo ϒ $\overline{oocd}$ to $\frac{oocd}{n}$ by *Conſ.* 2. 7:

2. *For* $\mathfrak{EI}$ *the abſciſſa.*

As *ob* to *eb* ſo *n* to $\frac{en}{o}$ = $\mathfrak{EI}$.

3. *For the ſide RS* ▭ *deficient or exceeding, from* Prop. 2. *and* 3.

As *n* to $\frac{oocd}{n}$ ſo $\frac{en}{o}$ to $\frac{ocde}{n}$ = RS.

Now the abſciſſa multiplyed by the *Lat. Rect.*

$\frac{en}{o}$ by $\frac{oocd}{n}$ gives ▭ *oecd.*

The ſum of theſe ▭ ▭ *oecd* † *eecd* in the hyperb. = □ $\mathfrak{IK}$ by *Prop.* 2. evidently = □ IK

The abſciſſa multiplyed by RS.

$\frac{en}{o}$ by $\frac{ocde}{n}$ gives ▭ *eecd.*

The difference of theſe ▭ ▭ *oecd* —— *eecd* gives in the ellipſis □ $\mathfrak{IK}$ by *Prop.* 3. evidently = □ IK.

Wherefore the lines $\mathfrak{IK}$ and IK, and the whole $\mathfrak{KL}$ and KL will be equal; and ſince the ſame thing is evident after the ſame way of all other lines of this kind *ad infinitum*, the elliptical and hyperbolical ſegments will be ſo alſo. Q. E. D.

CHAP. III.

Of Conoids and Spheroids.

Proposition XIV.

(α) *Archimed. Prop* 23. *and* 24. (*al.* 26. *and* 27.)

A Parablick Conoid (α) *is subduple of a Cylinder, and in sesquialteran reason* (*or as* $1\frac{1}{2}$) *of a cone of the same base and altitude.*

Demonstration.

Because in the parabola the □ AD (*Fig.* 137.) is to the □ SH, as BD to BH, *i. e.* as 3 to 1, and so to the □ TI as BD to BI, *i. e.* as 3 to 2, by *Prop.* 4. of this; it is evident that these squares of SH and TI and AD and consequently of the whole lines also S*h*, T*i*, AC, and the circles answering to them will be in arithmetical progression, 1, 2, 3; and moreover if there are new Bisections *in infinitum*, as the abscissa's so also the squares and circles of the ordinates, by vertue of the aforesaid fourth *Prop.* will always be in arithmetical Progression 1, 2, 3, 4, 5, 6, *&c.* It is evident that an infinite series of circles in the conoid, consider'd as its indivisibles, will be to a series of as many circles equal to the greatest AC, *i. e.* the conoid to the cylinder AF as 1 to 2, or as $1\frac{1}{2}$ to 3, by *Consect.* 9. *Prop.* 21. or *Consect.* 4. *Prop.* 16. *lib.* 1. but to the same cylinder AF the inscribed cone ABC is as 1 to 3, by *Prop.* 38. *lib.* 1. therefore the cylinder, conoid and cone are as 3, $1\frac{1}{2}$, and 1. Q. E. D.

Proposition XV.

(α) *Archim.* 29. *and* 30. (*al.* 32. *and* 33.)

THE half of (α) *any Spheroid, or any other segment of it is in subsesquialteran proportion to the cylinder, and double of the cone having the same base and altitude.*

Demon-

Demonstration.

Having divided the altitude BD (*Fig*. 138.) *e. g.* into three equal parts, because in the ellipse as well as in the circle the square of AD is to the square of SH as the ▭ GDB to the ▭ GHB, *i. e.* as 9 to 5, and so to the square TI as 9 to 8, by *Consect*. 1. *Prop*. 5. of this; and in like manner if you make new bisections, the squares (and consequently the circles) of the ordinates go on or decrease by a progression of odd numbers, as 36, 35, 32, 27, 20, 11, and so *ad infinitum*, the bisections being continued on; as we have shewn in the sphere and circumscribed cylinder *Prop*. 39. *lib*. 1. and it will necessarily follow here also (by vertue of *Consect*. 12. *Prop*. 21.) that the whole cylinder will be to the inscribed segment of the spheroid, as 3 to 2; and since the same cylinder is to the cone ABC as 3 to 1, also the segment of the spheroid will be to the cone as 2 to 1. Q. E. D.

Proposition XVI.

An hyperbolical Conoid (α) *is to a cone of the same base and altitude, as the aggregate of the ax of the hyperbola that forms it and half the Latus Transversum, to the aggregate of the said axis and Latus Transversum.*

(α) *Archim. Prop.* 27. *and* 28. (*al.* 30. *and* 31.)

Demonstration, containing also the Invention of this Proportion.

Make (in *Fig*. 139.) CE $= a$, EF $= b$, OE $= c$; then will CF $= a + b$. Since therefore
as CE to OE so CF to FQ

$$a \text{ —— } c \text{ —— } a+b \text{ —— } \frac{ac+bc}{a}$$

the □ EO will $= cc$ and □ FQ $= \dfrac{aacc + 2abcc + bbcc}{aa}$.

But as these squares so also are the circles of the lines EO and FQ to one another, by *Prop*. 32. *lib*. 1. and so the cone COP

will be as $\frac{acc}{3}$, and the cone CQR as $\frac{acc}{3} + bcc + \frac{bbcc}{a} + \frac{b^3cc}{3aa}$ (*viz.* by multiplying the third part of the altitude CF by the base FQ:) Having therefore substracted the cone COP from the cone CQR, there will remain the truncated cone QOPR $bcc + \frac{bbcc}{a} + \frac{b^3cc}{3aa}$, and from this solid truncated cone having further substracted the hollow truncated cone, which the space EHRP produced in the genesis of the conoid (and which according to *Consect.* 2. *Definit.* 9. is as bcc) there will remain the hyperbolical conoid $\frac{bbcc}{a} + \frac{b^3cc}{3aa}$, i. e. (by substituting now the values of the ax or abscissa EF, and of half the *Lat. Transv.* EC, and of the conjugate diam. OP, *&c.* found in the demonstrations of the preceding Chapter, *viz.* $\frac{ob}{2}$ for a, eb for b, and $\sqrt{oocd}$ for c or $oocd$ for cc) the hyperbolical conoid will come out $2eebocd + \frac{4e^3bcd}{3}$ i. e. $\frac{6eebocd + 4e^3bcd}{3}$. But the cone GEH (multiplying the third part of EF into the $\square$ GH, *i. e.* $\frac{1}{3}eb$ into $4oecd + 4eecd$) is as $\frac{4eebocd + 4e^3bcd}{3}$. Therefore the conoid is to the cone as $6eebocd + 4e^3bcd$ to $4eebocd + 4e^3bcd$, i. e. (dividing on both sides by $4eecd$) as $\frac{1}{2}ob + eb$ to $ob + eb$. Which was to be found and demonstrated.

SCHOLIUM.

IF any one had rather proceed herein by indivisibles, as in the precedent Prop. having divided the ax EF (*Fig.* 140.) again into three equal parts, and assuming the values of the lines determined in the hyperbola, *viz.* eb for the abscissa EF, ob for the transverse ax, $\frac{ocd}{b}$ for the *Latus Rectum*, $oecd + eecd$ for the square of the semiordinate FG, *&c.* the lowest and greatest circle of the diameter HG will be as $oecd + eecd$, and, if you make

as

as the *Latus Transv.* to the *Latus Rectum*, so the $\square\, DfE$
ob — $\frac{ocd}{b}$
made of $ob + \frac{2}{3}eb$ into $\frac{2}{3}eb$ (i. e. $\frac{2}{3}oebb + \frac{4}{9}eebb$) to a fourth; there will come out $\frac{2}{3}oecd + \frac{4}{9}eecd$ for the second circle of the diam. hg; and by the same inference (as ob to $\frac{ocd}{b}$ so $ob + \frac{1}{3}eb$ into $\frac{1}{3}eb$ to a fourth) for the third circle of the diameter HG $\frac{1}{3}oecd + \frac{1}{9}eecd$; so that these indivisibles [for which here and in the precedent also the partial circumscribed cylinders may be assumed] proceed in a double series of numbers, the first in a simple arithmetical progression 3, 2, 1, the latter in a duplicate Arithmetical progression of squares 9, 4, 1; and the same if you make further new bisections, will necessarily happen *ad Infinitum*, (the former numbers *e. g.* in the first bisection will be $\frac{6}{6}\ \frac{5}{6}\ \frac{4}{6}\ \frac{3}{6}\ \frac{2}{6}\ \frac{1}{6}\ oecd$ the latter $\frac{36}{36}\ \frac{25}{36}\ \frac{16}{36}\ \frac{9}{36}\ \frac{4}{36}\ \frac{1}{36}\ eecd$, &c.) it is manifest from the consectaries of *Prop.* 21. *lib.* 1. that the whole cylinder HK will in like manner be expressed by a double series of parts answering, in numbers to the indivisibles of the conoid made by any bisection, but in magnitude to the greatest of them all, and in the sum of its first series of parts will be to the sum of the first in the conoid, both being infinite, as 2 to 1 or 3 to $1\frac{1}{2}\ oecd$, by *Consect.* 9. of the said *Prop.* 21. and the sum of its latter to the sum of the former in the conoid will be as 3 to 1 $eecd$ and so the whole cylinder to the whole conoid as $3\,oecd + 3\,eecd$ to $1\frac{1}{2}\,oecd + eecd$ i. e. (dividing by ecd) as $3\,o + 3\,e$ to $1\frac{1}{2}o + e$ i. e. multiplying both sides by b) as $3\,ob + 3\,eb$ to $1\frac{1}{2}\,ob + eb$; and consequently the cone (which is $\frac{1}{3}$ of the cylinder) to the conoid as $ob + eb$ to $1\frac{1}{2}ob + eb$. Q. E. D.

CONSECTARY.

Hence also appears the proportion of the hyperbolick conoid to a cylinder of the same base and altitude, which we did not express in the *Prop. viz.* as the aggregate of the ax and half the *Latus Transversum* to triple the aggregate of the said ax and *Latus Transversum*.

CHAP.

CHAP. IV.

Of Spiral Lines and Spaces.

Proposition XVII.

(a) *Archim. Prop.*24.*de Spiral.*

THE (a) *first spiral space is subtriple of the first circle*, i. e. *as* 1 *to* 3.

Demonstration.

Having divided the circumference of the circle into (*Fig.* 141. *n.* 1.) three equal parts by lines drawn from the initial point, beginning from the first line BA, the line BC will be as 1, BD as 2, BA as 3, by *Consect.* 1. *Def.* 12. of this book, and consequently the sectors circumscribed about the spiral will be CB*c* as 1, DB*d* as 4, AB*a* as 9, by *Prop.* 32. *lib.* 1. and in like manner, if you make new bisections, the lines drawn from the point B to the spiral, will be 1, 2, 3, 4, 5, 6; but the circumscrib'd sectors, 1, 4, 9, 16, 25, 36; and so the circumscrib'd partial sectors *ad infinitum* will proceed in an order of squares, there being always as many sectors in the circle equal to the greatest of them. Therefore all the sectors that can be circumscrib'd *ad infinitum* about the spiral space, *i. e.* the spiral space it self (in which at last they end) to so many equal to the greatest, *i. e.* to the circle, is as 1 to 3, by *Consect.* 10. *Prop.* 21. *lib.* 1. Q. E. D.

CONSECTARY I.

SInce the first circle is to the second as 1 to 4 (*i. e.* as 3 to 12) by *Def.* 12. of this, and *Prop.* 31. *lib.* 1. and the first spiral space to the first circle as 1 to 3 by the present Prop. the same spiral space will be to the second circle as 1 to 12; and to the third by a like inference as 1 to 27, to the fourth as 1 to 48, *&c.*

CON-

CONSECTARY II.

THE first spiral line is equal to half the circumference of the first circle. For the lines or radii of the sectors, and consequently their peripheries or arches proceed in a simple arithmetical reason, as 1, 2, 3, 4, 5, 6, *&c.* while in the mean time the whole periphery of the circle contains so many arches equal to the greatest. Therefore the whole periphery of the circle is to an infinite series of circumscrib'd arches, *i. e.* to the spiral line it self, as 2 to 1, by *Consect.* 9. *Prop.* 21. *lib.* 1.

Proposition XVIII.

THE whole spiral (a) space comprehended under the second right line EA and the second spiral EGIA (see Fig. 141. *n.* 2.*) is to the second circle as* 7. *to* 12.

(a) *Archim. Prop.* 25.

Demonstration.

For having divided the circumference of the circle first into three equal parts, there will be drawn to the second spiral four right lines BE, BG, BI and BA being as 3, 4, 5, 6, and but only three sectors circumscrib'd, *viz.* GB*g*, IB*i* and AB*a*, which proceed according to the squares of the three latter lines, *viz.* 16, 25, 36, so that the sum is 77, while the sum of three equal to the greatest is 108, and so the one to the other (dividing both sides by 9) as 12 to $8\frac{5}{9}$. Having moreover bisected the arches and parts of the line BE, so that that shall be 6, the second BF will be 7, and so the other five 8, 9, 10, 11, 12; and the sectors answering to them (excepting the first) 49, 64, 81, 100, 121, 144, so that their sum shall be 559, while the sum of six equal to the greatest, *i. e.* the whole circle is 864, and so one to the other (dividing both by 72) as 12 to $7\frac{55}{72}$. In the other bisection of the arches and the parts of the line BE, so that the one shall be 12, the second 13, *&c.* to the thirteenth BA which will be 24, the sum of twelve sectors will be found to be 4250; and the sum of

as

as many equal to the greateſt 6912, and ſo the one to the other (dividing both ſides by 576) as 12 to $7\frac{218}{576}$ ſ. $\frac{109}{8}$

Therefore the proportion will be

I. In the firſt caſe 12 to $7 + 1 + \frac{1}{2} + \frac{1}{18}$ *viz.* $\frac{4}{72}$.
II. In the ſecond caſe 12 to $7 + \frac{1}{2} + \frac{1}{4} + \frac{1}{72}$ *viz.* $\frac{4}{288}$.
III. In the third caſe 12 to $7 + \frac{1}{4} + \frac{1}{8} + \frac{1}{288}$, *&c.*

The firſt and ſecond fractions thus decreaſing by $\frac{1}{2}$ the latter by $\frac{1}{4}$. Wherefore the proportion of the ſecond circle to the ſecond ſpiral ſpace will be as

$$12 \text{ to } 7 + 1 + \tfrac{1}{2} + \tfrac{1}{18}$$
$$- \tfrac{1}{2} - \tfrac{1}{4} - \tfrac{3}{72}$$
$$- \tfrac{1}{4}\ \&c. - \tfrac{1}{8}\ \&c. - \tfrac{3}{288}\ \&c.$$

By vertue of *Conſect.* 3. and 8. ⊟ ○ ⊟ ○
Prop. 21. *lib.* 1.
i. e. as 12 to 7. Q. E. D.

CONSECTARY I.

BEcauſe the ſecond circle is to the firſt ſpiral ſpace as 12 to 1, by *Conſect.* 1. of the preceding *Prop.* and to the ſecond ſpiral ſpace as 12 to 7, *by the preſent.* it will be to the ſecond ſpace without the firſt (*viz.* BCDEAIGE) as 12 to 6 *i. e.* as 2 to 1.

CONSECTARY II.

THerefore the ſecond ſpace ſeparately to the firſt is as 6 to 1.

CONSECTARY III.

SInce in the triſection of both theſe circles, firſt and ſecond, there ariſe ſix lines, and as many ſectors, *viz.* three lines BC, BD, BE, *i. e.* 1, 2, 3, to which there anſwer three arches in the ſame progreſſion within the ſecond circle, and alſo as many equal to its greateſt; therefore the ſum of all the unequal arches will be 21, but the ſum of the equal ones of both circles (each of which in the firſt are equivalent to 3, in the ſecond to 6) will be 27. Wherefore the ſum of both the Peripheries to the ſum of all the circumſcrib'd arches will be as

27

27 to 21, *i. e.* (dividing both sides by 9) as 3 to $2\frac{1}{3}$. Moreover bisecting the arches of the circles and the parts of the line BA, there will arise six circumscribed unequal arches within the first circle, which are as 1, 2, 3, 4, 5, 6, and as many within the second 7, 8, 9, 10, 11, 12; the sum of all which is 78, while the sum of as many equal ones on both sides is 108. Wherefore the one will be to the other, *i. e.* the sum of both the peripheries to twelve circumscribed arches taken together, is now as 108 to 78, *i. e.* (dividing both sides by 36) as 3 to $2\frac{1}{6}$. And making yet another bisection, the proportion will be found to be as 3 to $2\frac{1}{12}$, &c. and hence at length may be evidently inferr'd; that the sum of both the peripheries will be to the sum of all the arches circumscribible *ad infinitum*, i. e. to the whole helix as

$$3 \text{ to } 2 + \frac{1}{3} - \frac{1}{6} - \frac{1}{12} \text{ \&c.} = 0.$$

that is, as 3 to 2. Q. E. D.

CONSECTARY IV.

THerefore, since the periphery of the second circle is double of the first, that alone will be equal to the whole spiral.

CONSECTARY V.

THerefore, if the periphery of the second circle be 2, the periphery of the first will be 1, and the first spiral line $\frac{1}{2}$ by *Consect.* 2. of the anteced. *Prop.* wherefore the second spiral alone will be $1\frac{1}{2}$, and so the periphery of the second circle alone will be to the second spiral alone as 2 to $1\frac{1}{2}$ *i. e.* as 4 to 3; and to the first alone as 4 to 1.

SCHOLIUM I.

BUT as *Consect.* 4. may be also deduced after another way, *viz.* by comparing only the arches of the second circle with the correspondent circumscripts, but considering them as taken twice (because that circle is twice turned round while the whole helix or spiral is described) and finding in the first

trisection the proportion of double the second periphery to all the circumscripts as 12 to 7; and in the succeeding bisection as 12 to $6\frac{1}{2}$; in the second bisection as 12 to $6\frac{1}{4}$, *&c.* and at length by inferring, that the second periphery is double of all the arches circumscribible about the whole helix *ad infinitum*, that is to the helix it self.

$$\text{as } 12 \text{ to } 6 + \cfrac{1}{\cfrac{1}{2}\atop\cfrac{1}{4}\ \&c. = 0}\quad i.e. \text{ as } 12 \text{ to } 6;$$

and consequently the simple second periphery will be to the whole helix as 6 to 6: Thus the 5. *Consect.* may be separately had after the same manner, if instead of the first trisection, you only bisect; (*vid. Fig.* 141. *n.* 3.) for so in the first bisection the arches circumscribed about the second spiral line would be separately two semi-circles D*d*, 3 and A*a*, 4, (for as the line BC is one, BE, 2, BD, 3, BA, 4; so the arch described by the radius BD is 3 and described by the radius BA = 4,) and their sum 7; while the sum of two equal to the greatest is 8. In the second bisection (when BE is 4) BF and its arch is made 5, the arch BD 6, the arch BG 7, the arch BA 8, the sum 26; while the sum of so many quadrants equal to the greatest is 32 Thus in the third bisection the sum of eight Octants circumscrib'd about the second helix will be found to be 100, the sum of so many = to the greatest 128, *&c.* Wherefore the periphery of the second circle in the first case will be to the arches circumscrib'd about the second helix as 4 to $3 + \frac{1}{2}$; in the second as 4 to $3 + \frac{1}{4}$; in the third as 4 to $3 + \frac{1}{8}$, *&c.* and so to all the arches circumscribible *in infinitum*, i.e. to the second helix it self as

$$4 \text{ to } 3 + \cfrac{1}{\cfrac{1}{2}\atop{\cfrac{1}{4}\atop\cfrac{1}{8}\ \&c. = 0}}\quad i.e. \text{ as } 4 \text{ to } 3. \quad \text{Q. E. D.}$$

By the same method you may easily find the proportion of the third circle to the third spiral space, and of that periphery either to the whole spiral, or separately to the third, as will be evident to any one who trys.

I. For

I. *For the third spiral space.*

(*Fig.* 142)

BC 1	BF 4	BI 7	49
BD 2	BG 5	BK 8	64
BE 3	BH 6	BA 9	81

are the three first sectors circumscrib'd about the parts of the third helix. The sum of these three sectors is 194; and the sum of so many equal to the greatest 243. Therefore the first proportion of the one sum to the other will be as 243 to 194, *i. e.* (dividing both sides by 9) as 27 to $21\frac{1}{9}$.

In the first bisection there will be seven lines:

BH 12, BL 13	169
BI 14	196
BM 15	225
BK 16	256
BN 17	289
BA 18	324
Sum	1459

Sectors circumscribed about the parts of the third helix.

1459; while in the mean time the sum of as many equal to the greatest is 944, and so the second proportion as 1944 to 1459 *i. e.* (dividing both sides by 72) as 27 to $20\frac{19}{72}$.

In the second Bisection there will be thirteen lines, *viz.* BH 24, the rest 25, 26, *&c.* but the sum of the sectors, *i. e.* of the square numbers answering to the twelve latter will be found to be 11306; while in the mean time the sum of as many equal to the greatest will be 15552, so that you will have the third proportion of this sum to the other, *viz.* as 15552 to 11306, *i. e.* (dividing both sides by 576) as 27 to $19\frac{362}{576}$.

Therefore the I. proportion will be as 27 to $19 † 2 † \frac{1}{9}$ *i. e.*
to $19 † 2 † \frac{1}{2} † \frac{1}{18}$

II. —— as 27 to $19 † 1 † \frac{19}{27}$ *i. e.*
to $19 † 1 † \frac{1}{4} † \frac{1}{72}$

III. —— as 27 to $19 † \frac{362}{576}$ —— *i. e.*
to $19 † \frac{1}{2} † \frac{1}{8} † \frac{1}{288}$

Therefore the proportion of the third circle to the third spiral space will be

as 27 to 19 † 2 † $\frac{1}{2}$ † $\frac{1}{18}$
— 1 — $\frac{1}{4}$ — $\frac{3}{72}$ *i. e.* as 27 to 19.
— $\frac{1}{2}$ &c. — $\frac{1}{8}$ &c. — $\frac{3}{288}$ &c.
= 0. = 0. = 0. Q. E. D.

II. *For the third spiral line.*

If instead of the first trisection (as less commodious for the end proposed) you make use here also, as before, of bisection in the same figure, there will come out six lines from the point B to the helix, *viz.* B*m*, 1, BE, 2, B*n*, 3, BH, 4, B*o*, 5, BA, 6; to which there answer as many semicircular arches in the same progression, and to the greatest of the two as many equal to 2, 4, 6; so that the sum of the unequal ones is 21, and of the equal ones 24, and so the proportion of three peripheries together to all the circumscripts together will be as 24 to 21 (and dividing both by 6) as 4 to 3 $\frac{1}{2}$. In the second bisection the twelve unequal lines and arches make the sum 78, and as many equal to the greatest of the four will give the sum 96; so that the second proportion will be 96 to 78, *i. e.* (dividing both sides by 24) 4 to 3 $\frac{1}{4}$. In the third bisection the proportion will come out as 384 to 300, *i. e.* (dividing both sides by 96) as 4 to 3 $\frac{1}{8}$, *&c.* Therefore the proportion of the three circles together to the whole Helix will be as 4 to 3 † $\frac{1}{2}$
— $\frac{1}{4}$
— $\frac{1}{8}$ *&c.* = 0. *i. e.* as 4 to 3 or 12 to 9.

Q. E. D.

CONSECTARY VI.

NOW, if the periphery of the first circle be made 2, the second will be 4, the third 6, and consequently the sum 12; it will be manifest that the third periphery separately will be to the whole helix as 6 to 9, *i. e.* as 2 to 3.

CON-

CONSECTARY VII.

AND becauſe the ſecond periphery (which is 4) is equal to the firſt and ſecond helix together, by the above *Conſect.* 4. the remaining third ſpiral will be 5, and ſo the proportion of the third periphery to it as 6 to 5.

CONSECTARY VIII.

WHerefore the proportions of each of the peripheries to their correſpondent ſpirals will be in a progreſſion of ordinal numbers, *viz.* ſo that the latter of every two will denote the periphery of a circle, and the former an inſcribed ſpiral; and conſequently the ſpiral lines will be in an arithmetical progreſſion of odd numbers, and the peripheries of the circles in a progreſſion of even ones.

1———The firſt Spiral,
2———The firſt Periphery,
3———The ſecond Spiral,
4———The ſecond Periphery,
5———The third Spiral,
6 *&c.*—The third Periphery, *&c.*

SHCOLIUM II.

THE ſeventh Conſectary may alſo be eaſily deduced ſeparately this way: In the firſt biſection the line BA and its periphery is 6, the line B*o* and its periphery 5, the ſum of the circumſcribed Peripheries 11; the ſum of as many equal to the greateſt 12. Therefore the periphery of the third circle will be to the two circumſcripts as 12 to 11, *i. e.* as 6 to $5\frac{1}{2}$. In the ſecond biſection the four circumſcribed quadrants will be 12, 11, 10, 9, their ſum 42; and the ſum of four equal to the greateſt, *i. e.* the periphery of the third circle 48. Therefore the proportion is now as 48 to 42, *i. e.* (dividing both ſides by 8) as 6 to $5\frac{1}{4}$. Thus you will have the third proportion as 192 to 164, *i. e.* (dividing both ſides by 32) as 6 to $5\frac{1}{8}$. Wherefore the proportion of the third periphery to the third helix or ſpiral is

as 6 to 5 † $\frac{1}{2}$

— $\frac{1}{4}$

— $\frac{1}{8}$ &c.

= 0. *i.e.* as 6 to 5. Q. E. D.

CONSECTARY IX.

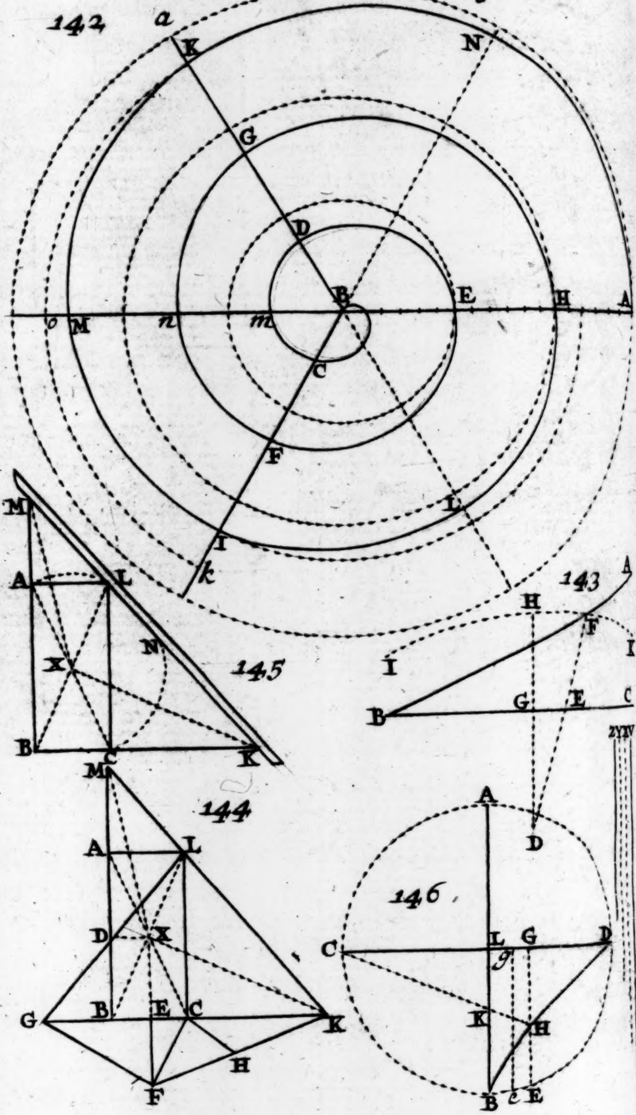
As Consect. 8. supplies us with a rule to determine the proportion of every spiral of every order to the periphery of the correspondent circle, *viz.* if the number of the order be doubled for the periphery of the circle, and the next antecedent odd number be taken for the spiral line; so what we have hitherto demonstrated supplys also another rule, to define the proportion of the spiral space in any order to its circle. For since the circles are in a progression of Squares 1, 4, 9, 16, &c. but the first circle is to the first space as 3 to 1 (*i.e.* a, 1 to $\frac{1}{3}$) by *Prop.* 17. and the second to the second as 12 to 7 (*i.e.* as 4 to $2\frac{1}{3}$) by *Prop.* 18. the third to the third as 27 to 19 (*i.e.* as 9 to $6\frac{1}{3}$) by *Schol.* 1. of this. And contemplating both these series one by another,

Of the circles, 1, 4, 9.

Of the spaces, $\frac{1}{3}$, $2\frac{1}{3}$, $6\frac{1}{3}$.

We see the numbers of the spaces are produced, if from the spuare numbers of the circles you substract their roots, and add to the remainder $\frac{1}{3}$. Wherefore, if, *e. g.* we were to determine the proportion of the fourth circle to the fourth spiral space; the square of 4 *viz.* 16 would give the circle; hence substracting the root 4, there wiill remain 12, and adding $\frac{1}{3}$ you would have the fourth spiral space $12\frac{1}{3}$; and in like manner the spiral space $20\frac{1}{3}$ would answer to the circle 25, &c. And that this is certain is hence evident, that if we multiply these numbers 16 and $12\frac{1}{3}$, also 25 and $20\frac{1}{3}$ by 3, that we may have those proportions in whole numbers, 48 and 37, 75 and 61, these are those very numbers *Archimedes* had hinted at in the *Coroll.* of *Prop.* 25.

CON-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CONSECTARY X.

NAY what we have now ſaid, is that very *Coroll.* comprehending alſo that 25*th. Propoſition, viz.* that a ſpiral ſpace of any order is to its correſpondent circle, as the rectangle of the ſemidiameters of this and the preceding circle together with a third part of the ſquare of the difference between both ſemi-diameters to the ſquare of the greateſt ſemidiameter. For, if *e. g.* the proportion of the third ſpiral ſpace to the third circle be required, ſince the ſemidiameter of this third circle is as 3, and the ſemidiameter of the ſecond precedent one is 2, and ſo the difference 1; the rectangle of 2 into 3 *i. e.* 6, together with $\frac{1}{3}$ of the ſquare of the difference will define the third ſpiral ſpace $6\frac{1}{3}$; ſince the third circle may be defined by the ſquare of the ſemidiameter of the greater, *viz.* by 9, and ſo in the reſt; as the numbers we have found ſhew, or further that may be found according to given Rules which may be here ſeen in the following Table.

Orders.	I	II	III	IV	V	VI	VII	VIII	XI	X
Circles.	1	4	9	16	25	36	49	64	81	100
The whole ſpaces, the preced. ones being included.	$\frac{1}{3}$	$2\frac{1}{3}$	$6\frac{1}{3}$	$12\frac{1}{3}$	$20\frac{1}{3}$	$30\frac{1}{3}$	$42\frac{1}{3}$	$56\frac{1}{3}$	$72\frac{1}{3}$	$90\frac{1}{3}$
Separate ſpaces the preced. ones being excluded.	$\frac{1}{3}$	2	4	6	8	10	12	14	16	18

CONSECTARY XI.

OUT of which table it is obvious to ſight, that the ſecond ſpace excluding the firſt is ſextuple of the firſt, as we have already deduced in *Conſect.* 2. *Prop.* 18. and the third ſeparate ſpace double of the ſecond, and the fourth triple of the ſame ſecond, and the fifth quadruple, and ſo onwards.

SCHO-

SCHOLIUM III.

AND this shall suffice for spirals, which comprehends not only the chief Theorems of *Archimedes* of spiral spaces, but also the chief of spiral lines (whereof *Archimedes* has left nothing.) If any should have a mind to carry on our method further, he may easily demonstrate after the same way what remains in *Archimedes*, and what Dr. *Wallis* in his Arithmetick of Infinites from *Prop.* 5. to the 38, and what others have done on this Argument.

CHAP. V.

Of the Conchoid, Cissoid, Cycloid, Quadratrix, &c.

Proposition XIX.

THE *first conchoid of* Nicomedes *Bbb (Fig.* 110) *on both sides of the perpendicular cDb approaches nearer always to the directrix or horizontal line AE, and yet will never coincide with it, altho' it be conceived to be produced on both sides* ad infinitum.

Demonstration.

For since only D*b* is perpendicular to AE, and all the rest *ab* are so much the more inclined to it by how much the more remote they are from the middle one D*b*, and all in the mean while are equal both to it and to one another, by *Def.* 13. it is evident that the points *b* and B will come so much the nearer to AE, by how much the farther they recede from the middle line D*b*. And yet because the lines BAC and *bac* are all right ones, whose points A, *a*, are in a right line AE it is equally as impossible that the point *b* or B, which is always in the conchoid should ever touch this right line, as it is impossible that the point C should be in it, by vertue of the aforecited *Def.* Q. E. D.

Propo-

Proposition XX.

YET *no other right line can be drawn between the directrix AE and the conchoid, but what will cut it if produced.*

Demonstration.

For if such a right line be made parallel to AE, as GH, and you make, as DI to IC so D*b* to a fourth, which will be greater than IC, as D*b* is greater than DI, and consequently, if making that an interval you draw the circular arch from C, it will necessarily cut the line GH *e. g.* in G. Drawing therefore C*a*G. you'l have as DI to IC so *a*G to GC, *i. e.* to that fourth proportional before found, by vertue of *Prop.* 34. *lib.* 1. but as DI to IC, so was also D*b* to the same fourth by Construc. Therefore *a*G and D*b*, which have both the same proportion to the same quantity, are equal; and consequently the point G is in the conchoid by virtue of *Def.* 13. and consequently the right line GH being produced will cut that produced also, on both sides, by the same reason. Much more will it cut it on either side if it be not parallel to the directrix AE, which is very obvious. Therefore no right line can be drawn between the conchoid, *&c.* Q. E. D.

CONSECTARY.

HEnce, besides other Problems, that may be very easily solved, which requires, having any rectilinear angle given ABC (*Fig* 143.) and a point without it, from that point to draw a right line DEF, so that part of it EF, which is intercepted between the legs of the angle, shall be equal to a given line Z. For if you draw the perpendicular DGH from the given point D through the nearest leg of the angle BC, and make GH equal to the given line Z, and from the center C at the interval GH describe the conchoid IHK, which will be necessarily cut by the other leg of the angle by vertue of the present *Prop.* *e. g.* in F, the line DF being drawn will

give the intercepted part = GH by the nature of the conchoid, and consequently = to the given line Z.

SCHOLIUM.

BY means of this Consectary *Nicomedes* solves that noble Problem of finding two mean proportionals, after this way, which we will here shew from *Eutocius*, but drawn into a compendium, and somewhat changed as to the order. Let two given lines AB and BC (*Fig.* 144.) between which you are to find two mean proportionals, be joined together at right angles, and divide both into two parts in D and E, and having compleated the rectangle ABCL, from L thro' D draw LG to BC prolonged; so that after this way GB may become = AL or BC: Having let fall a perpendicular from E cut off from C at the interval CF = AD the part EF, and having drawn FG make CH parallel to it; and lastly thro' the legs of the angle KCH draw the right line FHK, so that the part HK shall be equal to the line CF, by the preceding Consect. and also draw the right line KM from K thro' L to the continued line BA: All which being done, CK and AM will be two mean proportionals between AB and BC; which after our way we thus demonstrate: By reason of the similitude of the △ △ MAL and LCK

MA is to LC or AB as AL or BC to CK

b —— *eb* —— *c* —— *ec*

and moreover,

as MA to AD so GC to CK *i. e.* FH to HK

b —— $\frac{1}{2}eb$ —— $2c$ —— ec by reason of GF and CH being parallel, by *Consect.* 4 *Prop.* 34. *lib.* 1. therefore since HK is = AD = $\frac{1}{2}eb$, FH will be = A = b, and consequently MD = FK, *viz.* both b † $\frac{1}{2}eb$, and the square of both = bb † eb † $\frac{1}{4}eebb$ = □ EF † EK by vertue of the *Pythag. Theor.* Now if to these equal quantities you add the equal □ □ DX and EC = $\frac{1}{4}cc$, their sum, *viz.* □ MD † □ DX *i. e.* □ MX will be bb † ebb † $\frac{1}{4}eebb$ † $\frac{1}{4}cc$, equal to the sum of these, *viz.* □ EF † □ EC *i. e.* □ CF (by the *Pythag. Theor.* or EX by Construct.) † □ KX; whence these two things now follow: 1. That the lines MX and KX are equal. 2. If from those equal sums you take away the common quan-

tities

tities $\frac{1}{4}eebb + \frac{1}{4}cc$, the remainders will be equal, *viz.* $bb + ebb = ecc + eecc$; and (since the part taken away, *viz.* bb is manifestly to the other part taken away, *viz.* ecc as the remainder ebb to the remainder $eecc$, and the whole with the parts taken away and the remainders are in the same proportion by *Prop.* 26. *lib.* 1.) separately also bb will $=$ ecc and $ebb = eecc$. But from the latter equation it follows that

as eb to ec so ec to b by vertue of the 19. *Prop. lib.* 1.

AB to CK so CK to MA

and by the same reason it follows from the former Equation

as ec to b so b to c

CK to MA so MA to BC *i. e.* CK and MA are two mean Proportionals between AB and BC. Q. E. D.

From which deduction you have also manifest the foundation of that mechanical way, which *Hiero Alexandrinus* makes use of in *Eutocius*, *lib.* 2. *of the Sphere and Cylinder*, and which *Swenterus* has put into his practical Geometry *lib.* 1. *Tract.* 1. *Prop.* 23. when, *viz.* having joined in the form of a rectangle the given right lines AB and BC (*Fig.* 145.) and continued them at the other ends, he so long moves the ruler in L, having a moveable center, backwards and forwards, 'till XK and XM by help of a pair of compasses are found equal. To which, another way of *Philo*'s is not unlike, and flows from the same fountain, wherein, having made on AC a semicircle, the moveable ruler in L is so long moved backwarks and forwards, until LM and NK are found equal: Which seems to *Eutocius* to be more accommodated to practice, and easier to be perform'd by help of a ruler divided into small equal particles.

Proposition XXI.

IF *from any point of the other diameter in the generating circle e.g. from* G (*Fig.* 111. *n.* 1.) *you draw a perpendicular* GE *thro' the cissoid of* Diocles. *the lines* CG, GE, GD, *and* GH *will be continual proportionals.*

Demonstration.

For since GE and IF, as right sines, and also GD and IC as versed sines of equal arches by the Hypoth. are equal; you'l have as ID to IF (*i. e.* CG to GE) so IF to IC (*i. e.* GE to GD) *per n.* 3. *Schol.* 2. *Prop.* 34. *lib.* 1. But GD is to GH as ID to IF (*i. e.* as GE to GD) by the forecited *Prop.* 34. *lib.* 1. Therefore CG to GE, GE to GD, and GD to GH, are all in the same continual proportion. Q. E. D.

CONSECTARY.

HEnce it was easie for *Diocles* to find two mean proportionals *x* and *y* between two given right lines V and Z; (*Fig.* 146) for he made (having first described his curve DHB) as V to Z so CL to LK, and having drawn CKH to the curve, and thro' H the perpendicular GE, he had between CG and GH two mean proportionals GE and GD by vertue of the present *Prop.* when in the mean while CG the first would be to GH the last, as CL to LK, *i. e.* as the first V to the last Z given by vertue of the Constr. Therefore nothing remain'd but to make, 1. as CG to GE so V to X; and lastly, as GE to GD so *x* to *y*.

SCHOLIUM.

IT may not be amiss to mention here another way of finding two mean proportionals between any two given lines by the help of two Parabola's, which *Menechmus* formerly made use of, *viz.* by joining at right angles the given lines AB and BC (*Fig.* 147.) and prolonging them as occasion shall require thro' E and D; and then describing a Parabola about BE as its axis, so made that BC shall be its *Latus Rectum*, and in like manner describing another Parabola about BD as its ax, that shall have AB for its *Latus Rectum*, and that shall cut the former in F: Which being done, the semiordinate FE (or BD which is equal to it) being drawn to the point of Interfection F, will be the two mean proportionals sought. For by vertue of the fourth Consectary of *Prop* 1. of this Book, DF or BE

BE is a mean proportional between AB and BD, and in like manner EF or BD is a mean proportional between BE and BC, and consequently as AB to BE so BE to BD, and as BE to BD so BD to BC; Q. E. D.

To this way of *Menechmus* that of *Des Cartes* is not unlike, which he gives us *p. m.* 91. except only that instead of two Parabola's, he makes use only of one and a circle in room of the other: In imitation of whom *Renatus Franciscus Slusius* has since shewn infinite methods of doing the same thing by help of a circle, and either infinite Ellipse's or Hyperbola's, in his ingenious Treatise which he thence names his *Mesolabium.*

Proposition XXII.

ANY *semiordinate of the Cycloid as BF (Fig.* 148.) *or bf is equal to its corresponding Sine in the generating circle as BD, bd, together with the arch of that sine AD or Ad.*

Demonstration.

For the motion of the point A describing the semi-cycloid AFE, by *Def.* 11. is compounded of the motion of the orb (or wheel) B along the semi-circle ADC, and of the motion of the centre along the right line BC eqnal to CE, and consequently to the semi circle it self, or motion of the orb, Therefore as the point A moving to E by the motion of the orb (or wheel) moved or was carried from the diameter AC thro' the whole semi circle ADC 'till it came to AC again, and by the motion of the Centre passes thro' the whole space EG or CE, which is equal to the semicircular arch; thus the same A, when come to F, will have describ'd the quadrant AD, by moving from the diameter AC the quantity of the sine BD, and moreover by the motion of its centre (which is equal to the motion of the Orb) moves from AC the space of DF: And so the semiordinate BF will be equal to the arch AD and to its sine BD taken together; and in like manner the semiordinate bf will be equal to the arch Ad and its sine bd, &c. Q. E. D.

CON-

CONSECTARY I.

HEnce may be easily assign'd by help of the cycloid a right line equal to the semi-periphery or any given arch AD or A*d*; *viz.* CE double, or taken twice for the whole circumference, and single for the semi-periphery or half circumference, DF for the quadrant A*d*, *df* for the arch AD, *&c.*

CONSECTARY II.

WHerefore the quadrature of the circle may be geometrically obtain'd according to *Consect.* 2. of *Def.* 15. *lib.* 1.

CONSECTARY III.

IF you take B*e*, *be*, double of the sines BD, *bd*, *&c.* so that all the indivisibles *bd* taken together, may be to all the indivisibles *be* taken together as BD to B*e*, a curve described thro' the points *e* will be an ellipsis by *Prop.* 11. and the curvilinear space ADC*e*A will be equal to the semicircle ACDA.

CONSECTARY IV.

AND since DF (*i. e.* D*e* † *e*F) is equal to the quadrant DA, by vertue of the present Prop. BD † FG will be also equal to the quadrant (because the whole BG or CE is = to a semicircle) and consequently *e*F and FG will be equal; in like manner since *df* both above and below is equal to the arch *d*A, below *bd* † *fg* will be = to the remaining Arch *d*C: and above *bd* † *ef* (*i. e. df*) will be equal to the equal arch *d*A. Therefore *ef* above and *fg* below are equal, and (since the same may be shewn of all the indivisibles of the same sort throughout) the trilinear figure FGE will be equal to the trilinear *e*FA.

Propo-

Proposition XXIII.

THE *cycloidal space is triple of the generating circle* i. e. *the semi-cycloidal space AECA is triple of the semi-circle ADCA.*

Demonstration.

Since the parallelogram BCEG is equal to the whole circle by *Consect.* 2. of *Def.* 15. *lib.* 1. *i e.* to the semi-ellipse A*e*CA, by the present construction the Trapezium C*e*GE will be equal to the quadrant of the ellipse or the semi-circle. But the trilinear space FEG is ⵌ to the trilinear space FA*e*, by the fourth *Consect.* of the preced. therefore also the trilinear space A*e*CEA is equal to the semi-circle. Therefore the whole cycloidal space is equal to the three semi-circles. Q. E. D.

Or thus.

Since the whole parallelogram AE is equal to two circles and the semi ellipsis A*e*CA to one; the remaining space A*e*CEC to one circle, and its half A*e*GC to a semi-circle. But the trilinear space A*e*F is equal to the trilinear space FGE by *Consect.* 4. of the preced. Therefore the one being substituted in the other's place the trilinear space AFEC will be equal to the semi-circle: Therefore the remainder of the Parallelogram, *i e.* the cycloidal space AFECA will be equal to three semi-circles. Q. E. D.

SCHOLIUM.

TO these short Demonstrations, which we confess we owe for the most part to *Hon. Faber*, we will subjoin another somewhat more prolix, but yet not unpleasant, which we find in *Carolus Renaldinus, lib.* 1. *de Resol. & Compos. Math.* p. 299. But here we will give it the Reader more plain, and free from all Scruples, and likewise much easier. It is perform'd in these inferences, 1. That the right lined Parallelogram A*ba*B (*Fig.* 149 *n.* 1) is equal to the curvilinear space A*bda*BDA. 2. That

2. That as that is divided into two equal parts by its right lined diagonal A*a*, so likewise is this by the semi-cycloid A*aa*, so that the right lined triangle A*a*B is equal to the curvilinear space A*aa*BDA. 3. Therefore the one as well as the other is equal to the generating circle; and consequently, 4. If to this curvilinear space there be added the semi-circle ADBA the semi-cycloidal space A*aa*BA will be equal to three semi-circles. The first is evident, while if you take from the right lined parallelogram the semi-circle ADBA on the one side, and on the other add the semi-circle *abda*, there will arise the curvilinear Parallelogram. The third is evident from *Consect.* 2. *Def.* 15. *lib.* 1. Because the line B*a* is equal to the semi-periphery, which multiplyed by the diameter BC gives the area of the circle. The fourth is self evident; and so there remains only the second to be demonstrated, *viz.* That the curvilinear parallelogram is divided into two equal parts by the cycloid, *i.e.* that the external trilinear Figure A*adb*A is eqaal to the internal one A*aa*BDA; which may be thus shewn: Having divided the base B*a* into three equal parts, and drawn thro' them three semi-circles, and moreover the transverse right lines D*d* and E*e* thro' the intersections of the sem-icircles and the cycloid; it is certain from the genesis of the cycloid, by vertue of the Cons. of *Def.* 11. that as the right line *a* I is a third part of the whole *a*B, so the arch I *a* is a third part of the generating periphery, and by the same reason the arch 2*a* two thirds, and so the remaining arch *a* I I also $\frac{1}{3}$; insomuch that the first arch I *a*, and the last *a* I I, and consequently their right sines *af*, *ag*, and likewise their versed ones *f* I, *g* I I, are equal, and so the curvilined partial Parallelograms, both above and below, all upon equal bases, and of the same height (*viz.* the two linear ones *ae*2 I and *da* I I. I.) are equal among themselves, and so likewise the two pricked or pointed ones D*a* 2B and *ae b*I. Wherefore if now the base B*a* (*n.* 2) be conceived to be divided into six equal parts, and having drawn semi-circles, and transverse lines thro' their intersections with the cycloid, the arches

$\left.\begin{matrix} \text{I}a \\ \text{and V}a \end{matrix}\right\}$ will be $\frac{1}{6}$ $\left.\begin{matrix} 2a \text{ and} \\ \text{I V}a \end{matrix}\right\}\frac{1}{3}$, $\left.\begin{matrix} 3a \text{ and} \\ \text{III}a \end{matrix}\right\}\frac{1}{2}$ of a semi-circle,

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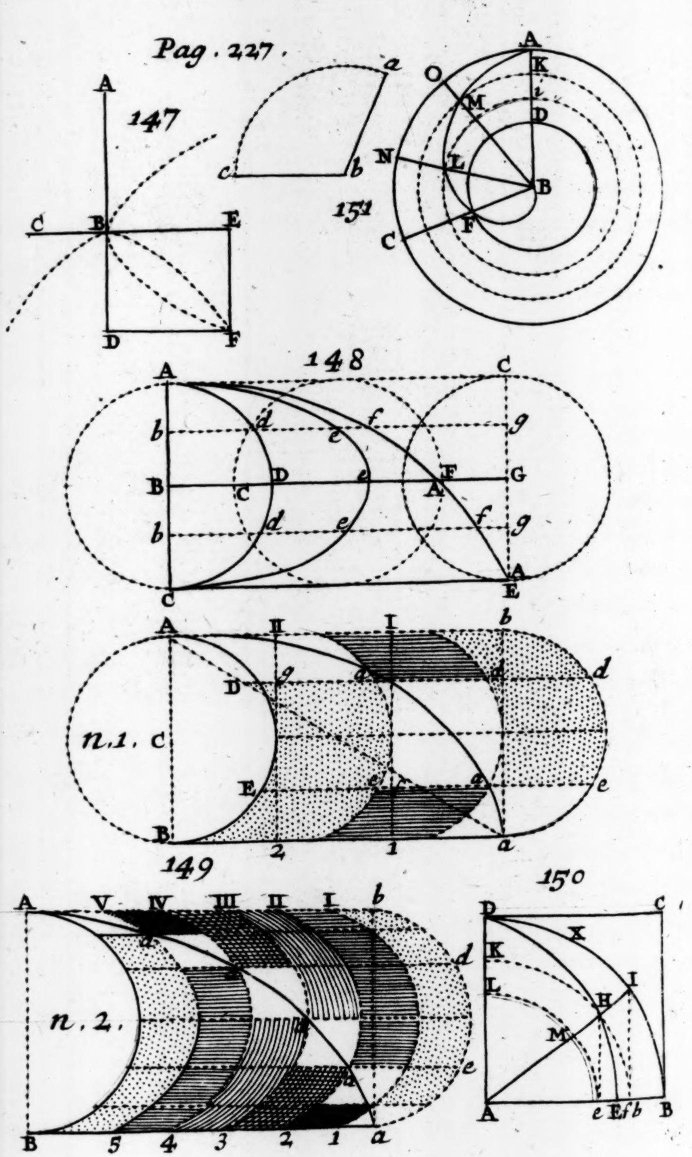


and so the versed sines of each, *i. e.* the altitudes of the corresponding parallelograms will be equal, and consequently the parallelograms of the internal and external trilinear space that are alike noted or signed will be equal to each other. Now this inscription of curvilinear parallelograms always respectively equal both in number and magnitude, since it may be continued in both the trilinear figures *ad infinitum*; it will evidently follow, that the trilinear figures themselves, whose infinite inscripts are always equal, will be likewise equal to one another.

Proposition XXIV.

THE base of the quadratrix AE (Fig. 150) and the semidiameter of the generating quadrant AD and the quadrant it self BD are in continual Proportion.

Demonstration.

For the quadrant DB is to the radius DA as the arch IB to the perpendicular He by *Consect.* 1. *Def.* 16 and I*b* is to He as A*b* to Ae by *Prop.* 34. *lib.* 1. But the arch IB (if it be conceiv'd to be less and less *ad infinitum*) will at length coincide with I*b*, as ending in the same moment in the point B, wherein He will end in the point E, and so A*e* will end in AE and A*b* in AB. Therefore at length DB will be to DA as IB (*i. e.* I*b*) to He, *i. e*, as A*b* to A*e*, *i. e.* as AB (or DA) to AE. Q.E. D.

SCHOLIUM I.

CLavius about the end of the sixth Book of *Euclid*, and others, demonstrate this indirectly by a deduction *ad Absurdum*, or concluding the opposite much after this manner: if DA or AB is not to AE as DB to DA, suppose it to be so to the greater A*f* or the less A*e*. In the first case therefore, because AB is to A*f* as DB to DA *per Hypoth. i. e.* as K*f* to A*f* the quadrant K*f* and the radius AB or DA will be equal. But as BD is to IB so is K*f* to H*f* by reason of the similitude of the arches; and as BD to IB so also DA (or = K*f*) to the sine He, by

X *Consect.*

Consect. 1. *Def* 16. Therefore the sine H*e* and the arch H*f* (to which the same K*f* bears the same proportion) will be equal; which is absurd. In the latter case, because AB would be to A*e* as DB to DA by the hypoth. *i. e* as L*e* to A*e*, the quadrant L*e* and the radius AB or DA would be again equal. But as BD is to IB so is L*e* to M*e* by reason of the similitude of the arches; and as BD to IB so also is DA (*i e* ⊐ L*e*) to H*e* by *Consect*. 1. *Def*. 16. Therefore the tangent H*e* and the arch M*e* (to which the same L*e* bears the same proportion) will be equal, which is again absurd. Wherefore BD is to DA, not as DA to a greater A*f* or a less A*e*; therefore as DA to AE. Q. E. D.

CONSECTARY I.

WHerefore it is evident from what we have deduced, if by means of the base of the quadratrix AE you draw a quadrant, the side of the quadratrix DA will be equal to it, and consequently double of the semi periphery, and quadruple of the whole periphery.

CONSECTARY II.

IT is evident also that you may obtain a right line equal to the quadrant DB of any given circle, if, having described a quadratrix, you make as AE to AD so AD to a third equal to the quadrant DB: Which third proportional taken four times will be equal to the whole periphery.

CONSECTARY III.

YOU may also obtain a right line equal to any less arch, if you make, as DA to H*e* so a third proportional found (*i e* the quadrant DB) to a fourth, by vertue of *Cons*. 1. *Def*. 16.

CONSECTARY IV.

THE quadrature of the circle therefore, by vertue of *Conſ.* 2. *Def.* 15. *lib.* 1. as likewiſe the triſection of an angle, by vertue of *Conſect.* 2. *Def.* 16. *lib* 2. may be Geometrically obtain'd, if the Quadratrix might be number'd among Geometrical Curves.

SHCOLIUM II.

CLavius was alſo of this Opinion in the book afore mentioned, who thought that if the quadratrix be excluded out of the number of geometrical curves, by the ſame reaſon you may alſo exclude the ellipſe, parabola, and hyperbola, ſince they as well as this are commonly deſcribed thro' innumerable points. But by that great Man's leave, we may deny this conſequence, by the ſame reaſon as *Des Cartes* has deny'd the converſe of it in his *Geom.* p 18. and 19. by vertue of which he ſuſpects the ancients took the conick ſections, *&c.* for mechanick or non-geometrick lines, becauſe they did the *ſpiral*, quadratrix, *&c.* for ſuch. But this is the difference between the deſcription of the quadratrix and the conick ſections thro' points, that all and every of the points of the conick ſections, relating to any given point of the axis, may be geometrically determin'd; but all the points of the quadratrix promiſcuouſly related to any point of the generating quadrant, cannot be geometrically determin'd, but only thoſe which reſpect ſome certain point, from which the quadrant may be divided into two arches of known proportion. For if, *e g.* in the quadrant BD the point X be given at pleaſure, it will be impoſſible by *Clavius*'s Rule to define a point of the quadratrix anſwering to it, becauſe the proportion of the arches DX and BX is unknown, and conſequently neither can a proportional ſection of the right line AD be made: Not to mention that the laſt point E (which is the primary and moſt neceſſary one to the quadrature) even by *Clavius*'s own confeſſion cannot be geometrically defined. We may paſs the like judgment on *Archimedes*'s ſpiral and ſuch like curves, which are conceiv'd to be deſcribed by two motions independent on one another; as will

will be manifest to any one who compares the genesis of the spiral with that of the quadratrix and what we have hitherto said. Whence neither will *Monantholius*'s trisection of a given angle (which he essays) by means of a spiral be enough geometrical; which in his Book *de Puncto, Cap.* 7. p. 24. he attempts to perform thus: To the centre of a described spiral and its first helical or spiral line BA (*Fig.* 151.) he applies the angle ABC equal to the given one *abc*; then having drawn circles thro' F and A where the legs of the angle cut the spiral, he divides the intermediate space DA into three equal parts in I, and K: And then thro' these points he draws circles cutting the helix in L and M; and lastly having drawn BLN, BMO, he easily demonstrates from the genesis of the spiral that the arches AO, ON, NC are equal. And so after the same manner not only any angle or arch, but the whole periphery may be geometrically divided into as many parts as you please; only supposing that this spiral line may be numbred among geometrical ones; as we have heretofore hinted that the cycloid, conchoid, cissoid, and logarithmical curve, *&c.* might be; and we have above sixteen Years ago declared our opinion for it in our *German* Edition of *Archimedes*; and now are therein confirm d by those celebrated Mathematicians *Leibnitz, Craige, &c.* who number lines of this kind, altho' they cannot be expressed by our common equations, among geometrical ones, notwithstanding the contrary opinion of *Des Cartes, &c.* because they admit of equations of an indefinite or transcendent degree, and are capable of a *Calculus* as well as others, tho' it be of a nature and kind different from that commonly used. See the *Acta Erud. Lips. ann.* 84. p. 234. and *ann.* 86. p. 292. and 294.

CHAP.

CHAP. VI.

The Conclusion, or Epilogue of the whole Work.

NOW we may at length underſtand what *Honoratus Fabri* delivers concerning the diſtribution of figurate magnitudes into certain Claſſes, in his *Synopſis Geom.* p. 57. and the following.

1. The firſt Claſſe contains elementary figures, or equal indiviſibles, ſuch as, 1. All Parallelograms, as the Square, Oblong, Rhombus and Rhomboid, the elements whereof are equal right lines, as in *Def.* 12. *lib.* 1. 2. Convex or concave Surfaces, the elements whereof are curve lines moved thro' right lines by a parallel motion; among which are chiefly reckon'd cylindrical ſurfaces, whereof ſee *Def.* 16. *lib.* 1. about the end. 3. Parallelepipeds, and among them the cube, whoſe indiviſibles are ſquares, or other Parallelograms. 4. Priſms made by the motion of a Triangle, Trapezium, or any Polygonous Body, along a right line, all the indiviſibles whereof are conſequently ſimilar and equal to the generating plane.

2. The ſecond Claſſe contains Figures whoſe Elements decreaſe in a ſimple arithmetical Progreſſion; ſuch are, 1. Triangles, as is evident from *Prop.* 37. *lib.* 1. 2. The circle, and its Sectors, as reſolvible into concentrick Peripheries according to *Conſ.* 1. and 3. of the aforecited *Prop.* 3. The Cylinder as reſolvible into concentrick cylindrick Surfaces, as its indiviſibles. 4. The Surface of a Cone, whoſe elements are circular Peripheries, and alſo of the Pyramid whoſe indiviſibles are ſimilar angular Peripheries every where increaſing in arithmetical Progreſſion. 5. The *Parabolick Conoid*, whoſe indiviſibles are Circles according to the proportion of the abſciſſa's in arithmetical Progreſſion, by vertue of *Prop.* 14. *lib.* 2. &c.

3. The third Claſſe contains elementary Figures increaſing in duplicate arithmetical Progreſſion; ſuch are, 1. The Pyramid and Cone, the firſt whereof may be reſolv'd into angular Planes, the ſecond into circular ones increaſing according to a ſeries of ſquare numbers; as is evident from *Prop.* 38. *lib.*

lib. 1. and its Consectary. 2. The trilinear parabolick Space, as defin'd *Prop.* 10. *lib.* 2. by the letters E*b* HK. 3. The Sphere, as far as it may be resolved into spherical concentrick Surfaces, every one whereof may be consider'd as a base, taking the semidiameter for the altitude. 4. The Cone, as resolvible into parallel conical Surfaces describ'd by the parallel indivisibles of the Triangle. 5. The remainder of a Cylinder after an Hemisphere of the same base and altitude is taken out, according to *Schol.* 1. of *Prop.* 39. *lib.* 1.

4. The fourth Classe would comprehend all magnitudes resolvible into elements or indivisibles increasing in triplicate, quadruplicate, &c. Arithmetical Progression; such we have not treated of, but may be found among Planes terminated by Curves of superior Genders; see *Fabri*'s Synopsis, p. m. 67.

5. The fifth Classe is of those Magnitudes, whose indivisibles decrease, proceeding from a square number by odd numbers, as, 36, 35, 32, 27, 20, 11, &c. such are, first, an *Hemisphere*, as is evident from *Prop.* 39. *lib.* 1. 2. An *Hemispheroid*, as in *Prop.* 15 *lib.* 2. 3. A *Semi-parabola*, as may be gather'd from the demonstration of *Prop.* 10 *lib.* 2. For since the indivisibles of the circumscrib'd trilinear figure *eb* are found in a duplicate arithmetical Progression, $\frac{1}{4}, \frac{4}{4}, \frac{9}{4}, \frac{16}{4}$, the indivisibles of the semi-parabola will necessarily be $\frac{16}{4}, \frac{15}{4}, \frac{12}{4}, \frac{7}{4}$, &c.

6. We may make a sixth Classe of those Magnitudes whose indivisibles decrease in a like Progression, not of the numbers themselves descending by odd steps from a given square, but of their roots, which are for the most part surd ones, such as is first, the *Semi-circle*, as is evident from *Prop.* 43. *lib.* 1. and by vertue of *Prop.* 5. *lib.* 2. and also the semi-ellipse, &c.

7. The seventh Classe comprehends those Magnitudes, whose Elements are in a Progression of a double series of numbers, as in the *Parabolick Conoid*, as may be seen in the *Scholium* of *Prop.* 16. *lib.* 2.

But, to omit the other Classes of Magnitudes of a superiour Gender, the consideration whereof these Elements either have not touch'd on, or only by the by; (which any one who pleases may see in *Faber*'s Synopsis, especially those which he comprehends under the sixth and seventh Classes, p. 70 and the following) about those we have here particularly noted, there remain

main only two things to be taken notice of. 1. That since in the first Classe we place Parallelograms and Cylinders, in the second Triangles, in the third Pyramids and Cones, in the fifth Hemispheres, in the sixth semi-circles, *&c.* We may with *Hon. Faber* call the first Classe, that of Cylindrical or Parallelogrammatick Figures; the second, the Classe of Triangular Figures; the third, of Pyramidals; the fifth, of Hemispherical Figures; the sixth of semicircular ones, *&c.* 2. That having ranged or reduced after this manner homogeneous Figures, or those of like condition, to a few Classes, their dimension, and consequently almost the whole business of measuring may be very compendiously reduc'd to a few Rules; whereof we will here give the Reader a short Specimen, in the following

CONSECTARYS.

I. THE dimension of Parallelogrammatick Figures, *i. e.* of those of the first Classe, may be had, by multiplying the whole base by the whole altitude: See *lib.* 1. *Def.* 12. *Cons.* 7. *Def.* 18. *Cons.* 6. *Def.* 16. *Cons* 3. and 4.

II. The dimension of Triangular Figures, *i. e.* of those of the second Classe, may be had by the multiplication of the whole Base by half the Altitude, or of half the Base by the whole altitude; [see *lib.* 1. *Def.* 12. *Consect.* 8. *Def.* 15. *Consect.* 2. *Def.* 18 *Consect.* 4. *lib.* 2. *Prop.* 14] and their Proportion is to their respective circumscribing Parallelograms, as 1 to 2; [see besides the *Prop.* already cited, *lib.* 1. *Prop.* 37. and its first *Consect.*]

III. The dimension of Pyramidals, *i. e.* of magnitudes of the third Classis, may be obtain'd by the multiplication of the Base by the third part of the of Altitude; [see *lib.* 1. *Def.* 17. *Consect* 3. and 4. and *Def.* 20. *Consect.* 1. &c.] and their Proportion to the corresponding Figures of the Classe of the same Base and Altitude is as 1 to 3. [see besides the *Prop.* already cited *Prop.* 38. *lib.* 1. and its *Cons. Prop.* 39. and *Schol.* 1. *lib.* 2. *Prop.* 10. *&c.*

IV. The

IV. The Proportion of Hemiſpherical Magnitudes, *i. e.* of the fifth Claſſis to correſponding ones of the firſt Claſſe of the ſame Baſe and Altitude is as 2 to 3; [ſee *lib.* 1. *Prop.* 39. *lib.* 2. *Prop.* 10. and 15.] and ſo their dimenſion may be had by multiplying theit Baſe by $\frac{2}{3}$ of their Altitude.

V. The Proportion of ſemi-circular Magnitudes, *i. e.* of thoſe of the ſixth Claſſe to ſo many correſponding ones of the firſt Claſſe of the ſame Baſe and Altitude cannot be expreſſed by whole numbers or by a ſmall fraction [ſee *lib.* 1. *Prop.* 43. and *lib.* 2. *Prop.* 11.] and conſequently their exact numeral dimenſion cannot be had.

A N

AN INTRODUCTION TO SPECIOUS ANALYSIS,

OR,

The New Geometry, chiefly according to the Method of *Des Cartes*,

But much facilitated by later Inventions, *&c.*

By *J. CHRIST. STURMIUS.*

THE
PREFACE
TO THE
READER.

SINCE *those ingenious Mathematicians of this present Age, which is now drawing to a Conclusion,* Vieta, Ougthred, Harriot, Cartes, Schooten, Beaune, Van Hudde, Heuraet, de Witte, *and* Slusius, *and several other Famous Men coeval with them, have by their Endeavours improved the* Algebra *of the Ancients, raised it to vastly an higher pitch, and brought it from Numbers to universal Symbols, and not only found the excellent uses of it in* Geometry *themselves, but also communicated them to others; almost all Countries have furnished us with some excellent Persons, who treading in the footsteps of their Predecessors, have endeavoured to advance it yet further: And even our Times are not without those of the highest rank, as* Wallis, Baker, Renaldinus, Mengolius, Huggens, Malbranch, Leibnitz, Craan, *and several others, who endeavour to promote this Science, deservedly reputed the very* Apex *of Humane Reason, and carry it more and more towards its utmost Perfection by dayly augmenting it with new and curious Inventions. But in the mean time, while these ingenious Men wholly busie themselves in promoting it, there are few found who condescend to explain the first Principles of it, and shew a ready way to young Beginners to arrive at the knowledge of those Inventions. It is not long since a certain Friend of mine, who for some time has publickly and successfully taught these Sciences,*

complain'd

complain'd to me by his Letters, of the want of a good Guide *to Specious Analysis, whereby he might instill the Principles of that admirable Art into his Auditors. To whose Desires being not just then at leisure to satisfie, I immediately after projected this Introduction, which at length you see finished, in the Form we now present you with it, wherein we have all along consulted to suit the Endeavours of young Beginners, as far as possible; as we thought our selves engaged by the Duties of our Professorship to do, & have compriz'd the Precepts of the Art in six or seven Pages, and accommodated Examples of every kind to illustrate them. Wherein if I have but indifferently accomplish'd my Design I shall not think my Labour lost. We here add it to our* Mathesis Enucleata, *both as being properly a part of it, and more especially, because this Introduction presupposes the Reader to be acquainted with the first Principles of Specious Computation, which we have therein laid down. And so we commit our Endeavours to the Perusal and Censure of the Candid Reader.*

INTRO-

INTRODUCTION TO SPECIOUS ANALYSIS.

THE *Analytick Art*, or *Specious Analysis*, is solely subbervient to finding of Theorems, and resolving Problems, by leading us from certain *Data* or given Quantities, into the knowledge of unknown and sought ones, by a Chain of certain and infallible Consequences: This admirable Artifice may be reduced to four Primary Heads, *viz.* *Denomination*, *Reduction*, *Equation*, and *Effection* (*if the Problem be a Geometrical one*) or *Construction*.

I. *DENOMINATION*.

BY Denomination is understood a preparatory imposition of Names peculiar to each Quantity, whereby every one of the Quantities given or sought, are denoted by one or more peculiar Letters of the Alphabet at pleasure, but with this (arbitrarious) difference, that known or given quantities are mark'd by the former Letters of the Alphabet, a, b, c, *&c.* and the unknown or sought ones by the latter, z, y, x, *&c.* But although this imposition of Names, is, as we have said, altogether arbitrarious, yet there often happens not a little facility to the Solution it self, by its being chosen as accommodate as possible to the conditions of the quantities given and sought; which any one will learn better by Use than Precepts: As we find that both Theorems may be demonstrated, and Problems resolved *e. g.* by an extraordinary Compendium, if we denote any reason of two given Homogeneous quantities by a and $e\,a$, b and $i\,b$, d and $o\,d$, *&c.* (*viz.* by expressing the Names of the Reasons by e, and i, and o, *&c.*) and continued proportionality

onality by a, ea, e^2a, e^3a, &c. and diſcontinued or diſcrete by $b\ i\ b$, $c\ i\ c$, $d\ i\ d$, or after the like manner, as we have done in our *Math. Enucl. Lib.* 1. *Cap.* 2, 3, 4, 7. and *Lib.* 2. *Cap.* 1. &c.

II. *EQUATION.*

HAving thus given each quantity its Name, and making no further diſtinction between the quantities given and thoſe ſought, but treating them all promiſcuouſly, and as already known, you muſt carefully ſearch into and diſcuſs all the Circumſtances of the Queſtion, and making various Compariſons of the quantities, by adding, ſubſtracting, multiplying, and dividing them, &c. 'till at length, which is the chief aim and deſign of it, you can expreſs one and the ſame quantity two ways, which is that we call an *Equation*: And you muſt find as many of theſe Equations, or Equalities of literal quantities, (as expreſſing the ſame thing) as there are ſeveral unknown quantities in the Queſtion, independent on each other, and conſequently denominated by ſo many different Letters, z, y, x, &c. But if ſo many Equations cannot be found, after having exhauſted all the Circumſtances of the Queſtions by one or two Equations; that is a ſign the other unknown quantities may be aſſumed at pleaſure: Which the Examples we ſhall hereafter bring will more fully ſhew.

But as here alſo (as likewiſe in all this Art) Ingenuity and Uſe do more than Rules and Precepts; yet we will here ſhew the principal Fountains, for the ſake of young Beginners, whence Equations, according to circumſtances obvious in the Queſtion, are uſually had. Theſe are partly Axioms ſelf evident, *E. g.*

That the whole is equal to all its parts taken together.

That thoſe quantities which are equal to one third, are equal among themſelves.

That the Products or Rectangles under the Parts or Segments, are equal to the Product of the whole.

Partly ſome univerſal Theorems that are certain and already demonſtrated, as,

Three

Three (α) continual Proportionals being proposed, the Rectangle of the Extremes is equal to the Square of the mean.

(β) Four being proposed, whether in continued or discontinued Proportion, the Product or Rectangle of the Extremes is equal to that of the Means.

And several others such like, which we have demonstrated in *Cap.* 2, 3, and 4. *Lib.* 1. of our *Mathesis Enucleat.* partly in the last place, some particular Geometrical Theorems already demonstrated, as *e. g.* that common Pythagorick one.

That in rightangled Triangles (γ) the Square of the Hypothenusa is equal to the two Squares of the sides.

That the Square of the Tangent of a (δ) Circle is equal to the Rectangle of the Secant and that Segment of it that falls without the Circle; the first whereof, we have demonstrated, *Lib* 1. *Math. Enuc. Def.* 13. *Schol.* and also *Prop.* 34. *Consect.* 8. also *Prop* 44. after various ways; to which may be numbred *Prop.* 34. with *Schol.* 11. *n.* 3. *Prop.* 37. and following, *Prop.* 45. and 46. also the 48. and several others in *Lib.* 1. *Math. Enucl.* and likewise *Lib* 2 *Prop.* 1, 2, 3, and several following. And as for Examples both of Denomination, and Equations found after various ways, you may see them hereafter follow, and some we will here give you by way of Anticipation.

III. *REDUCTION.*

An Equation thus found must be reduc'd, *i. e.* those two equal quantities, which for the most part are very much compounded of the quantities given and sought together, must be reduc'd to such a form, by adding or substracting something to or from each part, or multiplying and dividing by the same, *&c.* that the unknown or sought quantity alone, or its Square or Cube or Biquadrate, *&c.* may be found on the one side, and on the other the quantity express'd by meer given or known Letters, or affected with the unknown and sought ones; such are these Forms which follow, distinguish'd by their Names prefix'd to them,

(α) *Eucl.* 6. 17.
(β) *id.* 6. 16.
(γ) *id.* 1. 47.
(δ) *id.* 3. 36.

A ſimple Equation, $z = b$, or $y = \frac{ab}{c}$.

A pure Quadratick, $yy = ab$, or $xx = \frac{aa - bb}{2}$.

A pure Cubick, $z^3 = abc$, or $y^3 = \frac{a^3bb}{cd}$.

An affected Quadratick, $x^2 = -ax + b^2$; or,

$$y^2 = 2by + \frac{2abc}{d}.$$

An affected Cubick, $z^3 = az^2 + b^2z - c^3$ &c.

A Biquadratick, $y^4 = ay^3 + b^2y^2 - c^3y + d^4$ &c.

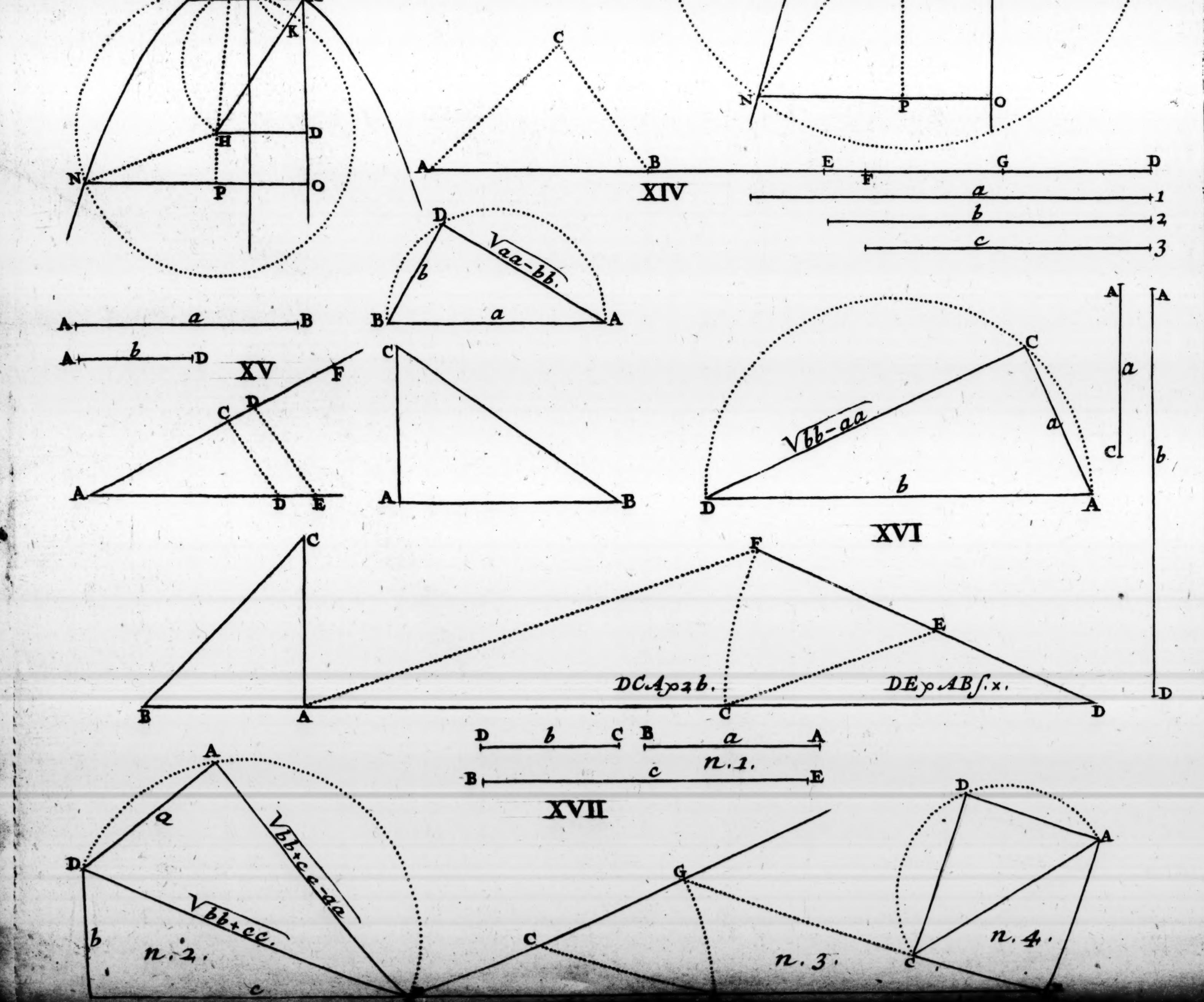
To one of which, or ſome other like them, when you have reduc'd your Equation firſt found, there are Rules in readineſs, whereby the Value of the unknown or ſought quantity z, or y, or x, may be either expreſſed in Numbers, if the Queſtion be an Arithmetical one, or geometrically determin'd if it be a Geometrical one: Which is that we call *Effection* or *Construction*.

Thus therefore the whole, or at leaſt the chief buſineſs of Analyticks, is converſant in finding a convenient or fit Equation: For Reduction is very eaſie, and conſiſting only in eaſie Operations and mere Axioms, as *e. g.*

If to equal quantities you add or ſubſtract equal ones, the Aggregates or Remainders will be equal;

If equal quantities are multiplyed or divided by the ſame, the Products or Quotients thence ariſing will be equal, &c.

IV. EFF


XIV
XV
XVI
XVII
$\sqrt{aa-bb}$.
$\sqrt{bb-aa}$
DCAp2b.
DEpABf.x.
n.1.
n.2.
n.3.
n.4.
$\sqrt{bb+cc}$.
$\sqrt{bb+cc-aa}$

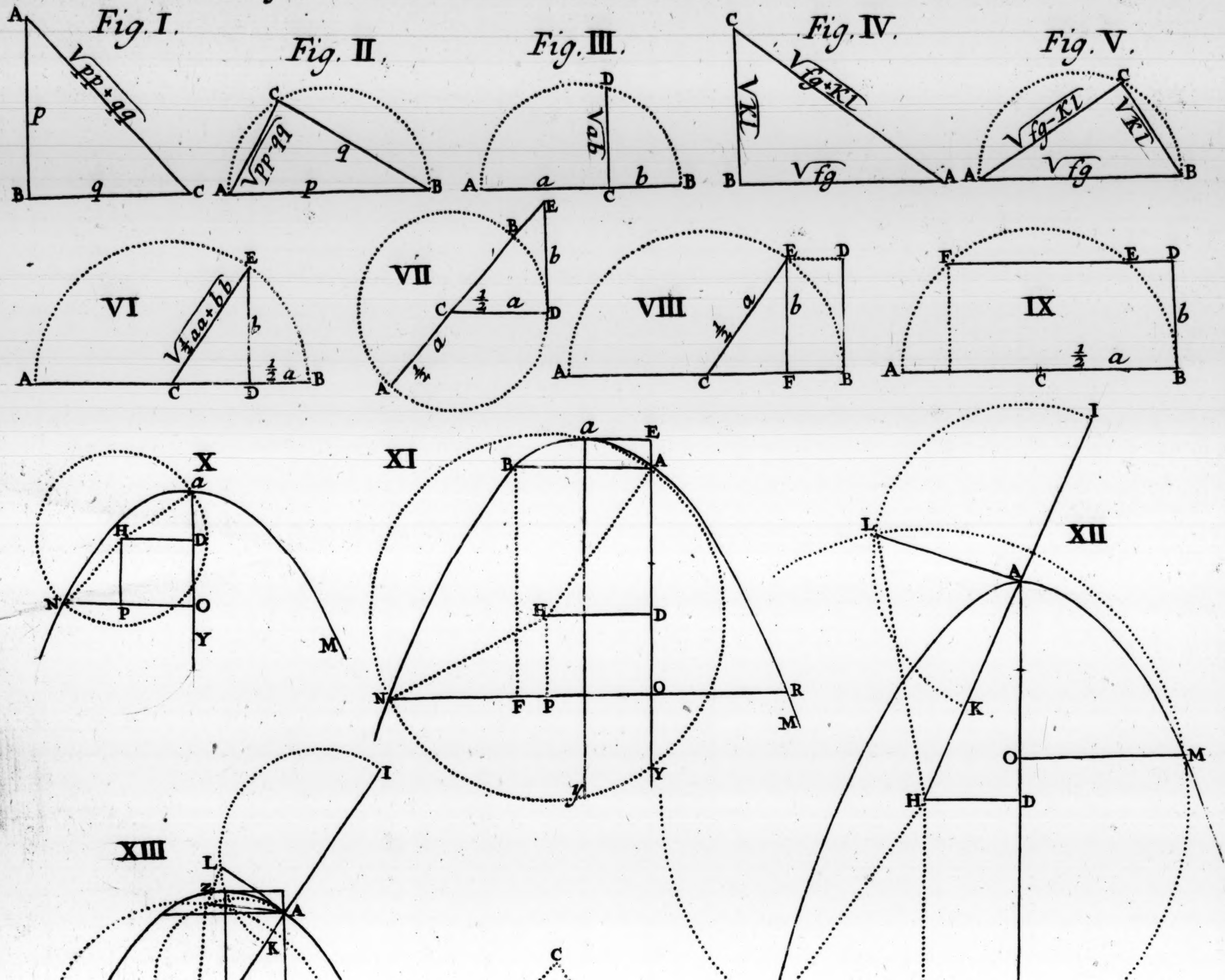


Fig. I.
Fig. II.
Fig. III.
Fig. IV.
Fig. V.
VI
VII
VIII
IX
X
XI
XII
XIII

.EFFECTION, or CONSTRUCTION.

1. *In simple Equations.*

Uppose $z = b$, the quantity b is sought.

If z be $= \frac{ab}{c}$.... as c to b so a to z,

or $x = \frac{aa}{b}$... make as b to a so a to x,

or $y = \frac{fb + fg}{b - 1}$ as $b - 1$ to $b + g$ so f to y,

or $z = \frac{fb - fg}{b + 1}$ as $b + 1$ to $b - g$ so f to z, &c. every where according to *n.* 2. Schol. 2. Prop. 34. Lib. 1. *Math. Enuc.*

If z be $= \frac{kl + mn}{r + s}$, the Resolution of it into Proportionals be more difficult, because neither of the Letters are found e in the Numerator. That therefore you may have *e. g.* wice, you must make as k to n so m to a fourth Proportio- which call p; then will, by vertue of *Prop.* 18. *Lib.* 1. $= mn$, and the proposed Equation be changed into this $= \frac{kl + kp}{r + s}$, to be now constructed from the 2*d*. Case.

Or if $y = \frac{kl + mn}{r - s}$, find a mean Proportional between k l, which call p; and between m and n, which call q, rding to *n.* 3. of the afore-cited Schol.; and the propo- d Equation, by virtue of *Prop.* 17. will be in this form: $\frac{pp + qq}{r - s}$. Make therefore in the right-angled △ (*Fig.* 1.) $= p$ and BC $= q$; and the □ AC by vertue of the Py- gorick Theorem, $= pp + qq$: Which since it must be di- ed by $r - s$, make further, by *Prop.* 18 as $r - s$ to the $pp + qq$, so is $\sqrt{pp + qq}$ to y, according to the afore- d.

4 In like manner if x be $= \frac{bg - mn}{c + d}$, make 1*st*. as b to m so n to a fourth which call k; and so putting bk for mn, the Equation will be reduced to the second Case under this form: $x = \frac{bg - bk}{c + d}$

Or thus: Find a mean Proportional between b and g, which call p, and between m and n, which call q; and the proposed Equation will be in this Form: $x = \frac{pp - qq}{c + d}$

Make therefore (in *Fig*. 2.) AB $= p$, and having on this described a Semi-circle, apply BC $= q$; then will, by vertue of *Schol*. 5. *Prop*. 34. $\square$ AC $= pp - qq$: Which since it must be divided by $c + d$, make farther,

as $c + d$ to $\sqrt{pp - qq}$, so $\sqrt{pp - qq}$ to x; all from the same Foundations, whence you have the Construction of the third Case.

5. If z be $= \frac{aabc}{ffg}$; make first as f to a, so a to a third Proportional m, and you'l have (putting fm for aa) $z = \frac{fmbc}{ffg}$, i. e. $\frac{mbc}{fg}$. Make secondly as f to m, so b to a fourth n, and by putting fn for mb you'l have $z = \frac{fnc}{fg}$, i. e. $\frac{nc}{g}$, wherefore thirdly you'l have as g to n so c to z.

6. If y be $= \frac{hll}{mm}$, make first as m to n, so l to a fourth, which call n, and by putting now mn for hl, you'l have $\frac{mnl}{mm}$ i. e. $\frac{nl}{m} = y$. Therefore you'l now have secondly, as m to n, so l to y by Case 2: So that the Construction of the fifth and sixth Cases is nothing but reiterations of the Rule of three, according to what we have often inculcated, *N*. 2. and 3. *Schol*. 2. *Prop*. 34.

2. *In*

2. *In simple Quadratick Equations.*

1.

IF $\left.\begin{array}{l} xx = ab \\ \text{or } y^2 = 1c \\ \text{or } z^2 = \frac{3}{4}dd \end{array}\right\}$ you'l have $\left\{\begin{array}{l} x = \sqrt{ab} \\ y = \sqrt{1c} \\ z = \sqrt{\frac{3}{4}dd} \end{array}\right.$ that is to a mean proportional between $\left\{\begin{array}{l} a \text{ and } b \\ 1 \text{ and } c \\ \frac{3}{4}d \text{ and } d \end{array}\right.$

and so the Construction will be had from *n.* 3. *Schol.* 2. *Prop.* 34. (see Fig. 3.)

2. If $\left.\begin{array}{l} y^2 = fg + kl \\ \text{or } x^2 = fg - kl \end{array}\right\}$ you'l have $\left\{\begin{array}{l} y = \sqrt{fg + kl} \\ x = \sqrt{fg - kl} \end{array}\right.$ make therefore on the one side the Right-angled Triangle ABC (*Fig.* 4) whose side

AB is = to a mean Proportional between f and g,
BC is = to a mean Proportional between k and l;

On the other a Right-angled $\triangle$ (*Fig.* 5) whose side AB is = to a mean Proportional between f and g, and the side BC = to a mean Proportional between k and l;

$\left.\begin{array}{l} \text{and on the one hand} \\ \text{the Hypothenusa,} \\ \text{on the other the side} \end{array}\right\}$ $\left.\begin{array}{l} \text{AC will} \\ \text{be the va-} \\ \text{lue of} \end{array}\right\}$ $\left\{\begin{array}{l} y \\ \\ x \end{array}\right.$ sought.

And all by vertue of the Pythag. Theor. and according to *Schol.* 5. *Prop.* 34. or the Consectarys of *Prop.* 44. See Fig 4. and 5.

3. If $z^2 = \frac{fhhkk}{fcc}$ i. e. $\frac{hhkk}{cc}$; extracting the Roots on both sides, z will $= \frac{hk}{c}$, and so be the second case of simple Equations.

4. If y^2 be $= \frac{fghk}{lm}$, make first as l to f, so g to a fourth which call n, and by putting ln for fg, you'l have $y^2 = \frac{lnhk}{lm}$ i. e. $\frac{nhk}{m}$. Make Secondly, as m to n, so h to a fourth, which call p; and by putting mp for nh, you'l have $y^2 =$ mpk

$\frac{mpk}{m}$ i. e. pk, and so the first case of the present Equations.

5. If $x^2 = \frac{qbcc + bccd + qbcd + bcdd}{fg + lm}$ in the first place the Rectangles fg and lm being turned into Squares, and collected into one Sum, make them $= nn$. Then (since cc and cd are multiplyed by $qb + bd$) in like manner qb and bd added make pp; and you'l have $x^2 = \frac{ppcc + ppcd}{nn}$. Thirdly (since pp is already multiplyed by $cc + cd$) having added $cc + cd$ into one Sum that they may *e.g.* make rr; x^2 will $= \frac{pprr}{nn}$, and so be the third case of the present Equations.

3. *In affected quadratick Equations.*

1. If zz be $= az + bb$, then will $z = \frac{1}{2}a + \sqrt{\frac{1}{4}aa + bb}$; which may be thus in short demonstrated *a priori*: Since $z^2 - az = bb$ *per Hypoth.* and that first quantity if it be added to $\frac{1}{4}aa$, it becomes an exact Square, the root whereof is $z - \frac{1}{2}a$; therefore $z^2 - az + \frac{1}{4}aa = bb + \frac{1}{4}aa$, and consequently $z - \frac{1}{2}a = \sqrt{bb + \frac{1}{4}aa}$; and lastly $z = \frac{1}{2}a + \sqrt{\frac{1}{4}aa + bb}$ or $\frac{1}{2}a - \sqrt{\frac{1}{4}aa + bb}$; which last Root is a false one and less than nothing, but yet gives you the proposed Equation back again as well as the former; as will be evident to any one who trys, *viz.* having transferr'd $\frac{1}{2}a$ on the other side, and so the two equal quantities $z - \frac{1}{2}a$ and $-\sqrt{\frac{1}{4}aa - bb}$ being squared. For here will come out $\frac{1}{4}aa + bb$ as well as if the radical Sign were affected with the Sign + because $-$ by $-$ gives +. Therefore $zz - az + \frac{1}{4}aa = \frac{1}{4}aa + bb$, and taking away on both sides $\frac{1}{4}aa$, $zz - az = bb$ i.e. $zz = az + bb$.

The value therefore of this Root will be had geometrically, by making (in *Fig.* 6.) $CD = \frac{1}{2}a$ and $DE = b$, that the Hypothenusa CE may be $\sqrt{\frac{1}{4}aa + bb}$; and moreover, drawing out on both sides CD, and at the interval CE describing a Semi-circle

mi-circle AEB: This being done, AD will be the value sought of the true Root z, and DB of the false one.

2. If y^2 be $= -ay + bb$, then will $y = -\frac{1}{2}a + \sqrt{\frac{1}{4}aa + bb}$; which again may thus appear: Since $y^2 + ay$ is $= bb$ *per Hypoth.* adding to both sides $\frac{1}{4}aa$, the first quantity will be an exact Square, and $y^2 + ay + \frac{1}{4}aa = \frac{1}{4}aa + bb$. Therefore the Roots will be also equal, *viz.* $y + \frac{1}{2}a = \sqrt{\frac{1}{4}aa + bb}$, and consequently $y = \sqrt{\frac{1}{4}aa + bb} - \frac{1}{2}a$ or $-\sqrt{\frac{1}{4}aa + bb} - \frac{1}{2}a$; which is a false Root.

The value of these Roots may be had geometrically, *viz.* of the true Root DB in *Fig.* 6. or BE in *Fig.* 7. and of the false one in the first AD, in the second AE.

3. If xx is $= ax - bb$, you'l have

$$x = \tfrac{1}{2}a + \sqrt{\tfrac{1}{4}aa - bb}$$
$$\text{or } \tfrac{1}{2}a - \sqrt{\tfrac{1}{4}aa - bb}.$$

Which may be demonstrated after the same way *a priori*, as the former Cases, *viz.* Since $x^2 - ax$ is $= -bb$, adding on both sides $\frac{1}{4}aa$, the former quantity will be an exact Square, *viz.* $x^2 - ax + \frac{1}{4}aa = \frac{1}{4}aa - bb$. Therefore the Root of the one $x - \frac{1}{2}a =$ to the Root of the other, *viz.* $\sqrt{\frac{1}{4}aa - bb}$, and adding on both sides $\frac{1}{2}a$, $x = \frac{1}{2}a + \sqrt{\frac{1}{4}aa - bb}$ which is one of the true Roots. Or $\frac{1}{2}a - \sqrt{\frac{1}{4}aa - bb}$ which in this case is also a true one. But the value of each may be obtain'd by making (*Fig.* 8.) CB $= \frac{1}{2}a$ and by erecting BD perpendicularly $= b$, and making the Semi-circle BEA, and drawing DE parallel to CB, and letting fall the Perpendicular EF: For thus CF will be $\sqrt{\frac{1}{4}aa - bb}$ and consequently AF, $\frac{1}{2}a + \sqrt{\frac{1}{4}aa - bb}$, and FB $\frac{1}{2}a - \sqrt{\frac{1}{4}aa - bb}$.

Or, with *Cartes*, making (*Fig.* 9) CB $= \frac{1}{2}a$ and BD $= b$, drawing DF parallel to CB, that FD may be one root and ED the other; as is manifest from the precedent Construction, and its Rule. See also another Deduction from *Cartes*'s Constructions, *Schol.* 1 *Prop.* 47. *Lib.* 1. *Math. Enucl.*

NB. 1. The ingenious *Schooten* has before shew'd this Method of demonstrating, and also of finding out these Rules in his Comment on the Geometry of *Des Cartes*, *p. m.* 163. and moreover deduces another ingenious Method for all the three Cases of these Equations, by taking away the second term in the

the Equation, *p.* 290, and the following, where we may make only this Remark concerning the third Case; that perhaps the Rule might be better deduc'd, if we make $x = \frac{1}{2}a - z$ rather than $x = z - \frac{1}{2}a$.

NB. 2. If any one has a mind to see the new Constructions of affected quadratick Equations of the Abbot *Catelan*, he may find them in *Acta Erud. Lips. Ann.* 1682. p. 86. and in the 27*th* Journal *des Scavans*. 1 *Dec.* 1681.

IV. *For Cubick and Biquadratick Equations both simple and affected, and also for all before mentioned, and consequently universally for all not exceeding the fourth Dimension.*

THE value of the unknown Quantity or Root may for any Case be determined by one general Rule, found out by Mr. *Thomas Baker* an English-man, occasioned by what *Des Cartes* had taught concerning this matter, *Lib.* 3. *Geom.* *p.* 85, and the following; but now very much perfected by this Rule, and made more simple. Now that this Rule may be the better comprehended by Learners, we will premise these following things.

1. That all Equations occurring under those Forms which we have before shewn in the Article of Reduction, or the like, must always for this purpose be so changed as to have all the terms or parts of the Equation both known and unknown, affected and not affected, brought over to one side promiscuously, and so on the other there will stand o or nought, as *e.g.* let $z - b$ be $= o$, or $y - \frac{ab}{c} = o$, or $z^3\ abc = o$, or

$y^3 - \frac{a^3bb}{ca} = o$, or $x^2 + ax - b^2 = o$; or

$y^2 - 2by - \frac{2abc}{d} = o$; or

$z^3 - az^2 - b^2z + c^3 = o$; or

$y^4 - ay^3 - b^2y^2 + c^3y - d^4 = o$ &c.

which also was usual to *Des Cartes* in *Lib.* 3.

2. In

2. In all Equations the known quantity or Co-efficient of the ſecond Term we will generally denote by the Letter p, that of the third Term by the Letter q, of the fourth by r, and the fifth (or abſolute Number) by S; according to *Cartes*, but with ſome little alteration: So that hence the Equations we have before been treating of, and all others like them (every where denoting the unknown quantity by x) may all be reduced to theſe forms:

$$x - p = 0$$
$$x^2 + px - q = 0$$
$$x^3 - px^2 - qx + r = 0$$
$$x^4 - px^3 + qx^2 - rx + S = 0, \text{ \&c. \&c.}$$

3. Theſe and the like Equations may either occur whole, or with all their Terms, as here, or depriv'd of one or more of them, as the following Examples will ſhew, where we will always put an Aſterisk in the place of the deficient Term.

$$x^2 \ast - q = 0$$
$$x^3 \ast\ast + r = 0$$
$$x^4 \ast\ast\ast - S = 0$$
$$x^3 \ast + qx \pm r = 0$$
$$x^4 \ast \pm qx^2 + rx \pm S = 0$$
$$x^4 + px^3 \pm qx^2 \ast \mp S = 0$$
$$x^4 \ast \ast + rx \pm S = 0$$
$$x^4 \mp px^3 \ast \ast \pm S = 0$$

4. The unknown quantity in any Equation has † as many diverſe Roots or Values, as the Equation has Dimenſions; which *Des Cartes* ſhews, *Lib.* 3. *Geom.* p. 69. at the ſame time evidently demonſtrating this, *viz.* that ſome of thoſe Roots may be falſe ones, *i. e.* leſs than nothing: Which from him we here ſuppoſe.

† N. *This is an error, as has been prov'd by Dr.* Wallis.

When therefore *Des Cartes* in his Conſtruction of Cubick and biquadratick Equations, *p.* 85, and the following, requires as a neceſſary Condition, the ejection of the ſecond Term in the given Equation, unleſs it were already wanting, and ſo was obliged to ſhew a way to eject it, with ſeveral other Preparations; and afterwards, when by help of his Rule delivered *p.* 91. he had found a way of finding two mean Proportionals, and dividing

dividing any given Angle into three equal parts, then he uses it for solving other solid Problems, or finding two mean Proportionals, or trisecting an Angle. But the general Rule of *Baker* has no need of these methods or helps, neither of the Ejection of the second Term, nor any other Preparation, but immediately shews us a way, by the help of a Circle and Parabola to find all the Roots of any given Equation, both true and false, whether the Equation want any term or not, and howsoever affected, after the way we will now, and perhaps a little more distinctly, shew.

1. It supposes with *Cartes* a Parabola NAM to be already described, (See *Fig*. 10. and 11.) whose *Latus Rectum* shall be L or 1, and its Axe ay; which *Des Cartes* only making use of, and never thinking of the other Diameters, was forced to take away the second Term of the Equation, *&c.* *Baker* therefore (strangely perfecting the *Cartesian* Geometry by this one thought) if the quantity p or second Term be in the Equation applys to the Ax ay (or draws an ordinate to it) $BA = \frac{p}{2}$ i.e. he erects at top of the Ax a on the right hand the perpendicular $aE = \frac{p}{4}$ and from E draws EAy parallel to the Ax ay; whereby he obtains the Diameter Ay sought.

2. Having made this Preparation, the whole business depends on this, to find the Center of the Circle to be described through the Parabola, which (by vertue of some arbitrary suppositions in the beginning) he always seeks on the left side of the Ax or Diameter, by help of two Lines AD / aD or b, and DH or d; *viz.* by placing the former upon the Ax from a to D, if p be wanting in the Equation, or upon the Diameter Ay from A to D if p be there; and letting fall from the point D the latter to aD or AD perpendicularly towards the left hand.

3. He shews how to find the quantity of either of these Lines (which is here very requisite) in any given Equation, by a certain general Rule (which he calls the Central Rule, because it alone helps to find the Center H) comprehended in these terms:

Part

The Eight half-sheet Plates that are to be folded in, and the single Leaf mark'd Page 89, *are to be placed in those Pages of the* Introduction *to* Specious Analysis *which the figures at the top of them direct to.*

A Synopsis *of Mr.* Baker's Clavis, *to be annexed to* Page 13, *of the* Introduction *to the* Specious Analysis.

Of Æquations.

Class. I.

II. 1. $x^2 * - q = 0$
2. $x^2 * + q = 0$ *Impossible.*

III. 1. $x^3 ** - r = 0$
2. $x^3 ** + r = 0$

1. $x^3 * - qx - r = 0$
2. $x^3 * - qx + r = 0$

IV. 3. $x^3 * + qx - r = 0$
4. $x^3 * + qx + r = 0$

$\frac{p}{2} = BA$

V. 1. $x - p = 0$
2. $x + p = 0$

1. $x^3 - px^2 * + r = 0$
2. $x^3 + px^2 * - r = 0$

VI. 3. $x^3 - px^2 * - r = 0$
4. $x^3 + px^2 * + r = 0$

1. $x^2 - px - q = 0$
2. $x^2 + px - q = 0$

Of Æquations.

1. $x^4 *** - S = 0$

1, 3: $x^4 * - qx^2 * \overline{+} S = 0$

2, 4: $x^4 * + qx^2 * \overline{+} S = 0$

1, 3: $x^4 ** - rx \overline{+} S = 0$

2, 4: $x^4 ** + rx \overline{+} S = 0$

1, 3: $x^4 * - qx^2 - rx \overline{+} S = 0$

2, 4: $x^4 * - qx^2 + rx \overline{+} S = 0$

5, 7: $x^4 * + qx^2 - rx \overline{+} S = 0$

6, 8: $x^4 * + qx^2 + rx \overline{+} S = 0$

$\frac{p}{2} = BA$

1, 3: $x^4 - px^3 ** \overline{+} S = 0$

2, 4: $x^4 + px^3 ** \overline{+} S = 0$

1, 3: $x^4 - px^3 + rx \overline{+} S = 0$

2, 4: $x^4 + px^3 * - rx \overline{+} S = 0$

5, 7: $x^4 - px^3 * - rx \overline{+} S = 0$

6, 8: $x^4 + px^3 * + rx \overline{+} S = 0$

1, 3: $x^4 - px^3 - qx^2 * \overline{+} S = 0$

2, 4: $x^4 + px^3 - qx^2 * \overline{+} S = 0$

Of Central Rules.

$\frac{1}{2} = b = AD$, $o = d = DH$.

$\left\{ \begin{array}{l} \frac{1}{2} + \frac{q}{2} \\ \frac{1}{2} \backsim \frac{q}{2} \end{array} \right\} = b = AD, o = d = DH$

$\left\{ \begin{array}{l} \frac{1}{2} = b = AD \\ \frac{r}{2} = d = DH \end{array} \right\}$

$\left\{ \begin{array}{l} \frac{1}{2} + \frac{q}{2} = b = AD \\ \frac{r}{2} = d = DH \end{array} \right\}$

$\left\{ \begin{array}{l} \frac{1}{2} \backsim \frac{q}{2} = b = AD \\ \frac{r}{2} = d = DH \end{array} \right\}$

$\frac{p}{4} = AE$ *in the* 5, 6, 7, 8 *Class.*

$\left\{ \begin{array}{l} \frac{1}{2} + \frac{p^2}{8} = b = AD \\ \frac{p}{4} + \frac{p^3}{16} = d = DH \end{array} \right\}$

$\left\{ \begin{array}{l} \frac{1}{2} + \frac{p^2}{8} = b = AD \\ \frac{p}{4} + \frac{p^3}{16} \backsim \frac{r}{2} = d = DH \end{array} \right\}$

$\left\{ \begin{array}{l} \frac{1}{2} + \frac{p^2}{8} = b = AD \\ \frac{p}{4} + \frac{p^3}{16} + \frac{r}{2} = d = DH \end{array} \right\}$

$\left\{ \begin{array}{l} \frac{1}{2} + \frac{p^2}{8} + \frac{q}{2} = b = AD \\ \frac{p}{4} + \frac{p^3}{16} + \frac{pq}{4} = d = DH \end{array} \right\}$

	$\left.\begin{matrix}4\end{matrix}\right\}$ $x^4+px^3-qx^2*+S=0$	$\frac{}{4}$ $\frac{}{16}$ $\frac{}{4}$
	$\left.\begin{matrix}8\end{matrix}\right\}$ $x^4*+qx^2+rx+S=0$	$\frac{}{2}$
$\frac{p}{2}=BA$	$\frac{p}{2}=BA$	$\frac{p}{4}=AE$ *in the* 5, 6, 7, 8 *Class.*
V. 1. $x-p=0$	$\left.\begin{matrix}1\\3\end{matrix}\right\}$ $x^4-px^3**\overline{+}S=0$	$\frac{1}{2}+\frac{p^2}{8}=b=AD$
2. $x+p=0$	$\left.\begin{matrix}2\\4\end{matrix}\right\}$ $x^4+px^3**\overline{+}S=0$	$\frac{p}{4}+\frac{p^3}{16}=d=DH$
1. $x^3-px^2*+r=0$	$\left.\begin{matrix}1\\3\end{matrix}\right\}$ $x^4-px^3+rx\overline{+}S=0$	$\frac{1}{2}+\frac{p^2}{8}=b=AD$
2. $x^3+px^2*-r=0$	$\left.\begin{matrix}2\\4\end{matrix}\right\}$ $x^4+px^3*-rx\overline{+}S=0$	$\frac{p}{4}+\frac{p^3}{16}\backsim\frac{r}{2}=d=DH$
VI. 3. $x^3-px^2*-r=0$	$\left.\begin{matrix}5\\7\end{matrix}\right\}$ $x^4-px^3*-rx\overline{+}S=0$	$\frac{1}{2}+\frac{p^2}{8}=b=AD$
4. $x^3+px^2*+r=0$	$\left.\begin{matrix}6\\8\end{matrix}\right\}$ $x^4+px^3*+rx\overline{+}S=0$	$\frac{p}{4}+\frac{p^3}{16}+\frac{r}{2}=d=DH$
1. $x^2-px-q=0$	$\left.\begin{matrix}1\\3\end{matrix}\right\}$ $x^4-px^3-qx^2*\overline{+}S=0$	$\frac{1}{2}+\frac{p^2}{8}+\frac{q}{2}=b=AD$
2. $x^2+px-q=0$	$\left.\begin{matrix}2\\4\end{matrix}\right\}$ $x^4+px^3-qx^2*\overline{+}S=0$	$\frac{p}{4}+\frac{p^3}{16}+\frac{pq}{4}=d=DH$
VII. 3. $x^2-px+q=0$	$\left.\begin{matrix}5\\7\end{matrix}\right\}$ $x^4-px^3+qx^2*\overline{+}S=0$	$\frac{1}{2}+\frac{p^2}{8}\backsim\frac{q}{2}=b=AD$
4. $x^2+px+q=0$	$\left.\begin{matrix}6\\8\end{matrix}\right\}$ $x^4+px^3+qx^2*\overline{+}S=0$	$\frac{p}{4}+\frac{p^3}{16}\backsim\frac{pq}{4}=d=DH$
1. $x^3-px^2-qx-r=0$	$\left.\begin{matrix}1\\3\end{matrix}\right\}$ $x^4-px^3-qx^2-rx\overline{+}S=0$	$\frac{1}{2}+\frac{p^2}{8}+\frac{q}{8}=b=AD$
2. $x^3+px^2-qx+r=0$	$\left.\begin{matrix}2\\4\end{matrix}\right\}$ $x^4+px^3-qx^2+rx\overline{+}S=0$	$\frac{p}{4}+\frac{p^3}{16}+\frac{pq}{4}+\frac{r}{2}=d=DH$
3. $x^3-px^2-qx+r=0$	$\left.\begin{matrix}5\\7\end{matrix}\right\}$ $x^4-px^3-qx^2+rx\overline{+}S=0$	$\frac{1}{2}+\frac{p^2}{8}+\frac{q}{2}=b=AD$
4. $x^3+px^2-qx-r=0$	$\left.\begin{matrix}6\\8\end{matrix}\right\}$ $x^4+px^3-qx^2-rx\overline{+}S=0$	$\frac{p}{4}+\frac{p^3}{16}+\frac{pq}{4}-\frac{r}{2}=d=DH$
VIII. 5. $x^3-px^2+qx+r=0$	$\left.\begin{matrix}9\\11\end{matrix}\right\}$ $x^4-px^3+qx^2+rx\overline{+}S=0$	$\frac{1}{2}+\frac{p^2}{8}\backsim\frac{q}{2}=b=AD$
6. $x^3+px^3+qx-r=0$	$\left.\begin{matrix}10\\12\end{matrix}\right\}$ $x^4+px^3+qx^2-rx\overline{+}S=0$	$\frac{p}{4}+\frac{p^3}{16}\backsim\frac{pq}{4}\backsim\frac{r}{2}=d=DH$
7. $x^3-px^2+qx-r=0$	$\left.\begin{matrix}13\\15\end{matrix}\right\}$ $x^4-px^3+qx^2-rx\overline{+}S=0$	$\frac{1}{2}+\frac{p^2}{8}\backsim\frac{q}{2}=b=AD.$
8. $x^3+px^2+qx+r=0$	$\left.\begin{matrix}14\\16\end{matrix}\right\}$ $x^4+px^3+qx^2+rx\overline{+}S=0$	$\frac{p}{4}+\frac{p^3}{16}\backsim\frac{pq}{4}+\frac{r}{2}=d=DH.$

N B. *The Sign* ∽ *denotes a dubious Case*, viz. *That either the Antecedent must be Substracted from the Consequent, or the Consequent from the Antecedent, according as the matter will bear.*

$$\begin{cases} 1.\ \dfrac{L}{2} \dagger \dfrac{p^2}{8L} \backsim \dagger \dfrac{q}{2L} = b = AD \text{ or } aD \\ 2.\ \dfrac{p}{4} \dagger \dfrac{p^3}{16L^2} \backsim \dagger \dfrac{pq}{4L^2} \backsim \dagger \dfrac{r}{2L^2} = d = DH. \end{cases}$$

4. This Rule as it ſtands here whole, only anſwers to thoſe ...ations wherein are all the Terms p, q, and r; and in the ...n time may alſo be eaſily accommodated to all other Caſes, ...y obſerving theſe things. 1. Whatever Term, or Quantities ..., r, be wanting in the propoſed Equation, that muſt alſo be ...ectively omitted, or put out of the general Central Rule, that ... remaining quantities may determine the ſpecial or particular ...tral Rule. 2. As for what belongs to the Signs, *viz.* whether ... $\backsim$ (which latter Sign denotes a dubious Caſe, either that the ... muſt be ſubſtracted from the latter, or contrary-wiſe, as ... matter will bear) muſt be put in the Central Rule, he ...es (α) that in the Rule you'l always have $\dagger \dfrac{r}{2L^2}$, unleſs ...en in the propoſed Equation p and r are affected with diverſe ...ns: (β) By what Sign ſoever in the propoſed Equation it ...pens that the quantity q is marked with, it muſt be noted ... the contrary one (altho' involv'd with other quantities) ...he Rule; as may be ſeen in the application of the Rule to ...ſpecial Caſes done by the Author himſelf for the ſake of Be...ners, and is exhibited in the Synopſis hereunto adjoyning, ...ich yet we have thought fit to give at the end of this Trea... much more contract as to the Central Rules, in a ſhort ...mpendium by way of Appendix.

5. By theſe Rules therefore, the quantities of the Lines aD AD and DH will be ſo determined, that the parts in the ... marked with the Sign † (taken either aggregately or ...ly) will be put downwards from a to A towards y, and on ... left hand of D; but the negative Parts, or thoſe affected ... the Sign—, will be cut off, on the one part above, on ...other on the right hand: Which being done the Center H ...be found.

6. From the Center H thro' the Vertex of the Ax a (if the ... D is found in the Ax) or in the other Caſe thro' the ...ex of the Diameter A, you muſt draw a Circle which by ...ng or touching the Parabola will deermine the Roots

ſought, if the Equation be not a Biquadratick *i. e.* has not the quantity S; otherwiſe another Point L or Z muſt be found, (*vid.* Fig. 12. and 13.) and a Circle deſcribed on the Radius HL or HZ, according to *Des Cartes* p. 86, and following, of his Geometry.

7. *Viz.* If you have — S, you muſt take on the Line H*a* or HA produced, on the one ſide AI = L or 1, and on the other $AK = \frac{S}{L^3}$, and deſcribing a Semi-circle on IK, draw AL perpendicular to AH, to obtain the point L. (ſee Fig. 12.) But if you have + S, then in another Semicircle deſcribed on AH, apply the Line AZ = to AL found, thereby to obtain Point Z, (ſee Fig. 13.)

8. A Circle therefore deſcribed from H through *a* or A, if S be wanting, but thro' L if there be — S, and thro' Z if + S, may touch or cut the Parabola either in 1, 2, 3 or 4 Points; from which if you let fall Perpendiculars to the Ax or Diameter, you will obtain all the Roots of your Equation both true and falſe.

9. And, 1. If in the Equation *p* be wanting and — *r* be there, the true Roots will be on the left ſide of the Ax, as NO, and the falſe ones as MO on the right ſide. 2. But if there be in the Equation *p* and — *p*, the true Roots will fall on the left ſide of the Diameter, and the falſe ones on the right; but if + *p*, on the contrary the true will be on the right hand and the falſe on the left.

10. But if the Circle neither touches nor cuts the Parabola in any point, it is a ſign that the Equation is impoſſible, and has no Root either true or falſe, but only imaginary ones. All which, how they may be found out, and that they are undoubtedly true, are demonſtrated *a poſteriori*, in an eaſie and plain way by the Author, wherefore we ſhall not give the Demonſtrations of them here; but remit the Reader, after he has made a little progreſs in this Art, to the Author himſelf.

11. Wherefore now, (omitting alſo in this place the Doctrine of the Compoſition of the plain and ſolid Geometrical *Loci*, or Places, which would ſerve for a Complement of the Analytick Art) we will ſhew the Practice of theſe Rules already delivered, premoniſhing only this from Mr. *Baker*, if the *Latus Rectum* be made Unity, that L in the Central Rules and

and all its Powers may be omitted, and ſo the Rules exhibited more compendiouſly, as we have already done in our Synopſis, and may be ſeen from the form of a general Central Rule hereunto annexed.

Part
1. $\frac{1}{2} + \frac{p^2}{8} \pm \frac{q}{2} = b = aD$ or AD

2. $\frac{p}{4} + \frac{p^2}{16} \pm \frac{pq}{4} \pm \frac{r}{2} = d = DH.$

To which Premonition of *Baker* we may alſo add this, if any given Line in the Problem it ſelf be taken for Unity, which may be often very commodiouſly done [as *a* in the former Problem p. 91. *Geom. Cartes*, and the Line NO in the latter, and *a* again in the Equation p. 83. the laſt line] and then the ſame Line alſo may be taken for the *Latus Rectum* of the Parabola to be deſcribed, if we have a mind to make uſe of this Compendium for abbreviating the Central Rules. For otherwiſe if we would conſtruct all Problems, as *Baker* rightly aſſerts we may, by only one Parabola, we ſhall fall often into very tedious Prolixities.

SOME EXAMPLES OF SPECIOUS ANALYSIS,

In each kind of Equations.

I. *In Simple Equations.*

PROBLEM I.

Having the sum of any two sides given for forming a Triangle ABC, to find each of the sides, and form the Triangle.

Supppose *e. g.* three Lines given in *Fig.* 14. the first $=$ $AB + AC$ in the Triangle sought, the second $= AB + BC$, the third $= BC + AC$, to find each of the sides *e. g.* to find AB, which being known, the rest will be so also.

SOLUTION.

1. *Denomination.* Make $AC + AB = a$; $AB + BC = b$; $BC + AC = c$; $AB = x$; then will $AC = a - x$, and $BC = b - x$, and so the Denomination be compleat.

2. *Equation.* Now if the values of the two last Lines BC and AC be added into one Sum, which we had before given; you'l have this Equation $a + b - 2x = c$.

3. *Reduction.* By adding on both sides $2x$, you'l have $a + b = c + 2x$; and substracting from both sides c .. $a + b - c = 2x$; and dividing both sides by 2 ... $\frac{a + b - c}{2} = x$.

4. The

4. The *Effection* or *Geometrical Construction*, which the Equation thus reduced will help us to

Join AE $= a$ and ED $= b$ in one Line AD, and from this backwards cut off DF $= c$; and divide AF which remains into two equal parts in B, and you'l have AB the first side of the Triangle to be formed; and BE will give the other side AC, which substracted from ED, will leave GD $=$ to the third side BC; of which you may now form the Triangle ABC.

5. *A general Rule for Arithmetical Cases.* Add the two former Sums, and from the Aggregate substract the third Sum; half the Remainder will give the side AB common to the two former Sums. For an Example take this Question: There are three Towns of ancient *Hetruria*, viz. *Forum Cassii* (which the Letter A denotes in △ ABC) *Sudertum* (B) and *Volsinii* (C) which are at this distance one from another; if you go from *Volsinii* to *Forum Cassii* and thence to *Sudertum*, you mst go 330 Furlongs; from *Forum Cassii* to *Sudertum* and thence to *Volsinii* there are 306 Furlongs; lastly, from *Sudertum* to *Volsinii* and thence to *Forum Cassii* 272 Furlongs. How far is each Town distant from each other.

PROBLEM II.

In a right-angled Triangle ABC, having given the Base AB, and the difference of the Perpendicular AC and the Hypothenusa BC to find the Perpendicular and Hypothenusa, and form the Triangle.

Make *e. g.* the Base AB (*Fig.* 15.) and the difference of the Perpendicular and Hypothenusa BD, to find the Perpendicular AC; which being known, the Hypothenusa AC will be known also, if the given difference be added to the found Perpendicular.

SOLUTION.

1. *Denomination.* Make AB $= a$, BD $= b$, AC $= x$; then will BC $= x + b$.

2. *Equation* by the Pythagorick Theorem,

$xx + aa = xx + 2bx + bb$, viz. the two Squares of the Sides to the Square of the Hypothenusa.

3. *Reduction*

3. *Reduction.* Substracting from both sides xx, you'l have $aa = 2bx + bb$; and moreover by substracting also bb, $aa - bb = 2bx$; and dividing by $2b$

$$\frac{aa-bb}{2b} = x.$$

4. *Effection* or *Geometrical Construction.* Having described upon the given Base AB a Semi-circle, apply therein the given difference BD, and draw AD, whose Square is $= aa - bb$. Since this must be divided by $2b$, make, as AE $= 2b$ to AD $= \sqrt{aa - bb}$, so AD $= \sqrt{aa - bb}$ to AC the Perpendicular sought. To which if you add CF $=$ BD, you will have AF $=$ to the Hypothenusa sought BC; which will come of course together with the whole Triangle sought, if the found Perpendicular AC be erected at right Angles on the given Base AB.

5. *The Rule for Arithmetical Cases.* From the square of the given Base substract the square of the given difference, and divide the Remainder by the double difference; and you'l have the Perpendicular sought. *E. g.* suppose the Base $=$ 20 foot, and the difference between the Perpendicular and Hypothenusa 10.

PROBLEM III.

IN the right angled Triangle ABC, having given the side AC and the sum of the other side AB and the Hypothenusa BC, to find the other side and the Hypothenusa separately, and form the Triangle. Suppose the given side (that is to be) AC (*Fig.* 16.) and the sum of the other sides AD, to find the side AB, which being known the Hypothenusa BC willl be known also.

SOLUTION.

1. *Denomination.* Make AC $= a$, AD $= b$, AB $= x$, then will BC $= b - x$.

2. *Equation.* $xx + aa = bb - 2bx + xx$ and substracting xx.

3. *Reduction.*

3. *Reduction.* $aa = bb - 2bx$; and adding $2bx$,
$aa + 2bx = bb$; and substracting aa,
$2bx = bb - aa$; and dividing by $2b$,

$$x = \frac{bb - aa.}{2b}$$

4. The *Effection* or *Construction* is like the former, and so will be manifest only by inspecting that Scheme.

5. *The Arithmetical Rule.* From the square of the given sum substract the square of the given side, and divide the Remainder by double the given sum; and youl l have the other side, and substracting that from the given sum, you have the Hypothenusa also. *E. g.* let one side be 15, and the sum of the other two 45.

PROBLEM IV.

Having given the Perpendiculars and sum of the Bases of two right-angled Triangles having equal Hypothenuses, to find the Bases separately, and form the Triangles. Suppose *e. g.* to form the Triangle ABC (see *Fig.* 17.) you have given the Perpendicular AB, and for the other △ ADC, the Perpendicular CD, and the given sum of the Bases BE, to find the Bases singly, *viz.* the less for the greatest Perpendicular, and the greater AD for the less Perpendicular.

SOLUTION.

1. *Denomination.* Make AB $= a$, CD $= b$, the sum BE $= c$; make the lesser Base BC $= x$; the greater AD will $= c - x$.

2. *Equation.* Since the Hypothenuses of the two Triangles are supposed equal, the two □ □ AB $+$ BC, *i. e.* $xx + aa$ will be $=$ to the two □ □ AD $+$ CD, *i. e.* $bb + cc - 2cx + xx$.

3. *Reduction.* By taking away therefore xx and adding $2cx$, $aa + 2cx$ will $= bb + cc$; and further taking away from both sides aa, $2cx = bb + cc - aa$; and dividing both sides by $2c$, $x = \frac{bb + cc - aa.}{2c}$

4. *Geome-*

4. *Geometrical Construction.* Join the Lines b and c in CD and BE at right Angles, (*n.* 2.) and the square of the Hypothenusa DE will $= bb + cc$. Upon this Hypothenuse having described a Semi-circle, apply therein the Line AD, and the square of AE will $= bb + cc - aa$. Which, since it must be further divided by $2c$, make (*n.* 3.) as BF $= 2c$ to BG $= \sqrt{bb + cc - aa}$, so BG to BC, the lesser Base sought.

5. *The Arithmetical Rule.* From the sum of the squares of the lesser Perpendicular and the sum of the Bases substract the square of the greater Perpendicular, and the Remainder divided by the double sum of the Bases, will give the lesser Base. *E. g.* make AB 76, CD 57, and BE 114.

PROBLEM V.

Having given the Perpendiculars of two right-angled Triangles standing on the same given Base, to find the Segments of the Hypothenuses. E. g suppose the common given Base be AB (*Fig.* 18.) and the Perpendicular of one Triangle AD of the other BC; to find the segments of the Hypothenuses, cutting one another geometrically.

SOLUTION.

If a Geometrical Solution be required there is no need of any Analysis; for having erected perpendicularly on the common given Base AB the given Perpendiculars AD and BC, the Hypothenuses AC, BD being drawn immediately, exhibit their Segments EA, EB, EC, ED. But if it be to be done arithmetically by a general Rule, then an Analysis will be necessary.

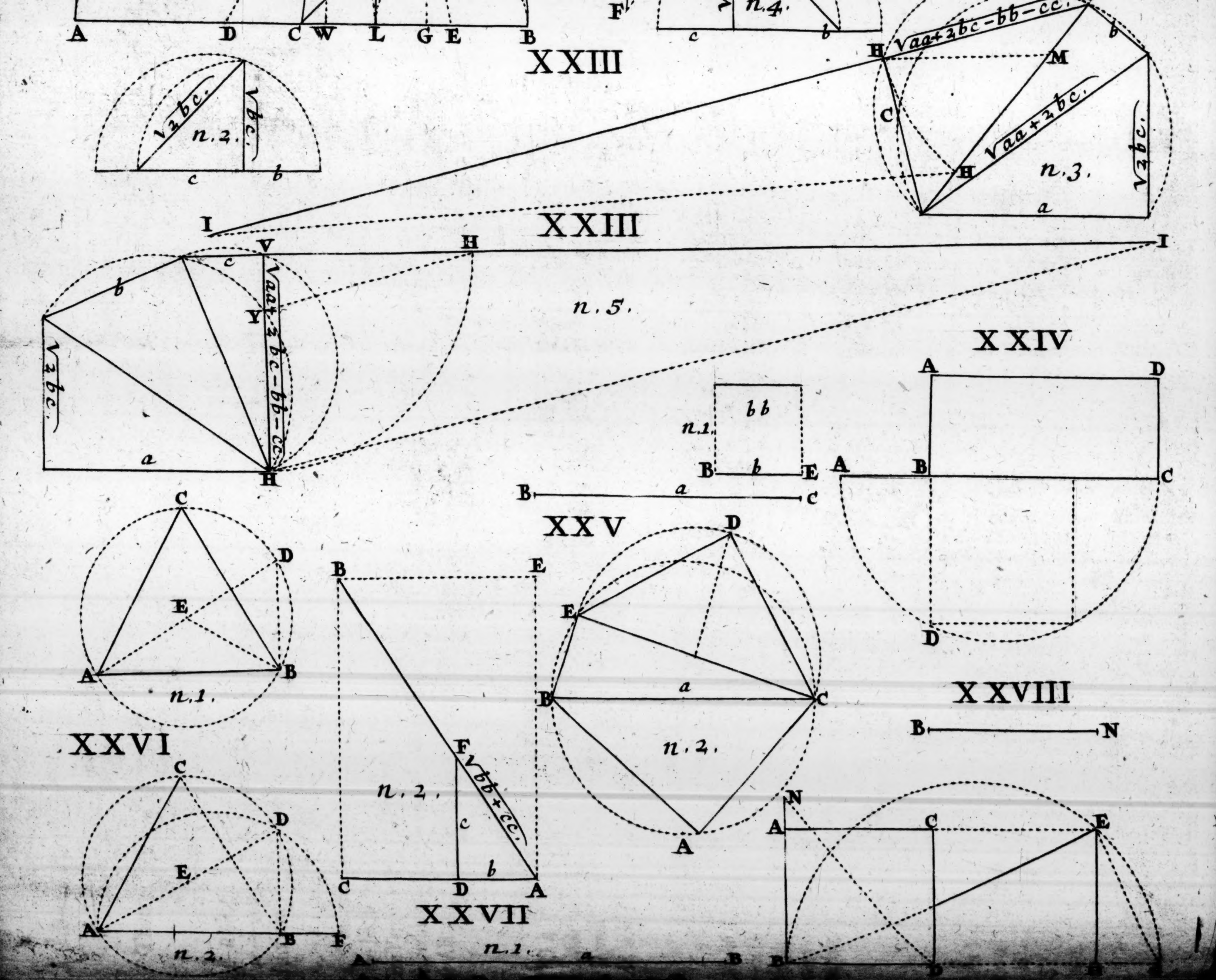
1. *Denomination.* Make the common Basis AB $= a$, BC $= b$, AD $= c$; and, having found AF, all the rest may be had (for as AB to BC so AF to FE; which being given, you have also GD and HC, and consequently also DE, CE, *&c.*) make AF $= x$, then will BF or HE $= a - x$.

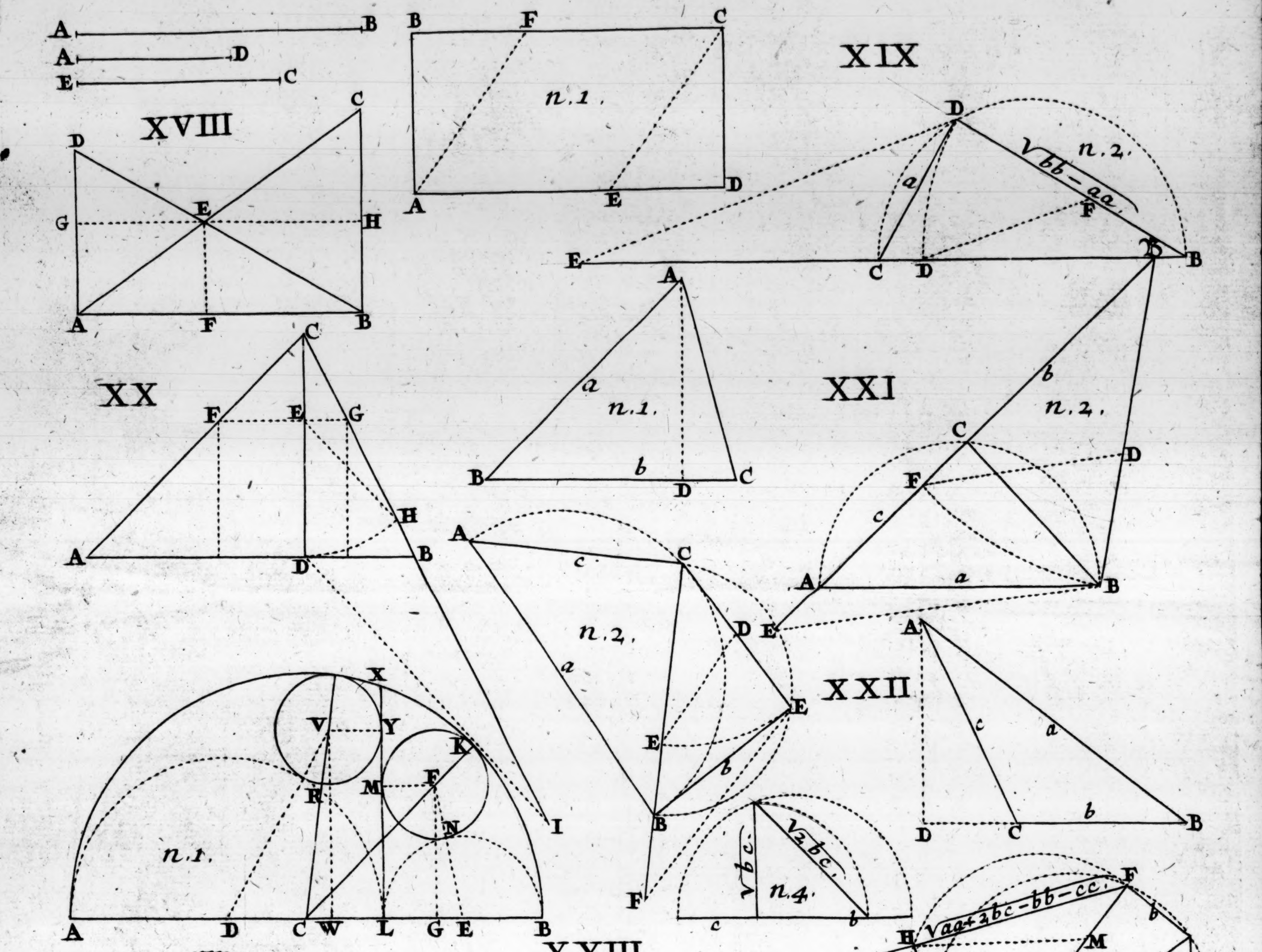
2. The *Equation* from FE found twice.

1. As AB to BC so AF to FE.

$$a \text{ --- } b \text{ --- } x \text{ --- } \frac{bx}{a}$$

2. As


XXIII
n. 2.
n. 3.
n. 4.
n. 5.
$\sqrt{2bc}$.
$\sqrt{bc}$.
$\sqrt{aa+2bc}$.
$\sqrt{aa+2bc-bb-cc}$.
XXIV
n. 1.
bb
XXV
n. 1.
n. 2.
XXVI
n. 1.
n. 2.
XXVII
n. 1.
n. 2.
$\sqrt{bb+cc}$.
XXVIII


XVIII
XIX
n.1.
n.2.
$\sqrt{bb-aa}$
XX
XXI
n.1.
n.2.
XXII
n.2.
n.1.
n.4.
$\sqrt{bc}$
$\sqrt{2bc}$
$\sqrt{aa+2bc-bb-cc}$

2. As BA to AD so BF to FE.

$$a - c - a - x - \frac{ac - cx}{a}.$$

Therefore $\frac{bx}{a} = \frac{ac - cx}{a}.$

3. *Reduction.* Multiplying both sides by a you'l have $bx = ac - cx$; and adding on both sides cx, $bx + cx = ac$ and dividing both sides by $b + c$, $x = \frac{ac}{b+c}$.

4. *The Arithmetical Rule.* Multiply the common base by the least Perpendicular, and divide the Product by the sum of the Perpendiculars; and you'l have the lesser segment of the base, which being given you'l have all the rest. *E g.* suppose AB = 10, BC = 9, AD = 6.

PROBLEM VI.

TO *inscribe a Rhombus in a given Oblong*, i. e. *having the sides of the Oblong AB and BC given, (Fig. 19.) to find the Segment BF or DE, which being cut off, the remainder FC or AE will be the side of the Rhomb sought.*

SOLUTION.

Make AB = a, BC = b, BF = x: FC or FA will be = $b - x$ (so far the *Denomination.*) Therefore the square of FA, which is $bb - 2bx + xx$ will be = $aa + xx$, *viz.* to the two squares of AB and BF (so far the *Equation*:) and substracting from both sides xx, $bb - 2bx = aa$; and

$bb = aa + 2bx$, and

$bb - aa = 2bx$; and

$\frac{bb - aa}{2b} = x$. (so far the *Reduction.*)

The Geometrical Construction. Having described a semicircle on BC (*n* 2.) apply CD or AB, and the □ BD will be $bb - aa$. Which since it must be divided by $2b$, make, as BE = $2b$ to BD = $\sqrt{bb - aa}$, so BD to BF sought, and be cut off the side of the Oblong BC (*n.* 1.)

 The

The Arithmetical Rule. From the ſquare of the greater ſide ſubſtract the ſquare of the leſſer, and divide the remainder by double the greater ſide, and the quotient will give the Segment BF ſought. *E. g.* ſuppoſe AB $=$ 4, and BC $=$ 8.

PROBLEM VII.

TO *inſcribe the greateſt ſquare poſſible in a given Triangle,* *i. e.* having given the heighth of the Triangle CD (*Fig.* 20) and the Baſe AB, to find a portion of the altitude CE, which being cut off there ſhall remain ED $=$ FG.

SOLUTION.

Make the baſe AB $= a$; the altitude CD $= b$, CE $= x$; then will ED or FG $= b - x$.

By reaſon of the ſimilitude of the Triangles ABC and FGC you'l have as AB to CD ſo FG to CE.

$$a - b - b \cdot x - x$$

Therefore the Rectangles of the means and extremes will be equal, *i e.* $ax = bb - bx$; and adding on both ſides bx, $ax + bx = bb$, and dividing by $a + b$, $x = \frac{bb}{a+b}$.

Conſtruction. Upon the ſide of the Triangle CB produced, make CH $= b$, and HI $= a$, ſo that the whole Line ſhall be $a + b$. And having joined ID and parallel to it HE drawn from H, the part CE will be cut off, which is that ſought.

For as CI to CD ſo CH to CE,

$a+b - b - b - x$ according to the ſecond caſe of ſimple Effections.

Arithmetical Rule. Square the given heighth of the Triangle, and divide the Product by the ſum of the baſe and altitude; and the quote is the part to be cut off CE. *E. g.* ſuppoſe CD $=$ 10, and AB $=$ 15.

PROBLEM VIII.

IN *an acute-angled Triangle having all the ſides given, to find the Perpendicular that ſhall fall from the Vertex on the Baſe,* i. e. having given AB, AC, BC (*Fig.* 21.) to find AD or

or BD (for having found the one you may easily find the other) *Coroll. Prop.* 13. *Lib.* 2. *Eucl.*

SOLUTION.

If there be only required a Geometrical Construction of this Problem, there will be no need of any Analysis; for having formed a Triangle ABC of the three given sides, you need only let fall the perpendicular AD from the Vertex A, which would determine the Segment BD. To find the general Arithmetical Rule, which is the general Corollary of *Euclid*, or if any one for exercise sake had rather determine the Perpendicular DA by the segment of the Base BD, than the latter by the former, the Analysis will proceed thus:

Make AB $= a$, BC $= b$, AC $= c$, BD $= x$; then will CD be $= b - x$. Wherefore by the Pythagorick Theorem □AD $= aa - xx$, and by the same reason the same □AD $= cc - bb + 2bx - xx$. Therefore $ax - xx = cc - bb + 2bx - xx$; and by adding on both sides xx, $aa = cc - bb + 2bx$; and by transferring $cc - bb$, $aa - cc + bb = 2bx$, and dividing by $2b$, $\frac{aa - cc + bb}{2b} = x$.

The Arithmetical Rule. Substract the square of the lesser side from the Sum of the □□ of the base and greater side, and the remainder divided by double the base will give its greater segment: If the □ of the greatest side be substracted from the sum of the other squares, &c. you will have the less segment CD.

Geometrical Construction Having described a semi circle upon AB (*n.* 2.) apply therein AC, and the □ BC will $= aa - cc$, and continuing AC to 𝔅, till C𝔅 be = CB (*n.* 1.) the □ of 𝔅B will $= aa - cc + bb$; which since it is to be divided by $2b$ make as 𝔅E $= 2b$ to 𝔅B $= \sqrt{aa - cc + bb}$, so 𝔅F = 𝔅B to 𝔅D the segment sought = BD *n* 1.

PROBLEM IX.

IN *an obtuse angled Triangle having the three sides given to find the Perpendicular let fall from the Vertex to the Base being continued*: i. e. Having given AB, BC, AC (*Fig.* 22. *n.* 1.) to find AD or CD (for the one being found the other will readily be so also) *Coroll Prop.* 12. *Lib.* 2 *Eucl.*

SOLVTION.

What we premonished about the former Problem, we understand to be premonish'd here also. For the rest make here also AB $= a$, BC $= b$, AC $= c$, CD $= x$; then will BD $= b + x$: Wherefore by the Pythagorick Theorem the $\square$ AD will $= cc - xx$, and by the same Theorem the same $\square$ AD $= aa - bb - 2bx - xx$.

Therefore $cc - xx = aa - bb - 2bx - xx$; and adding to both sides $cc = aa - bb - 2bx$; and transposing cc and $-2bx$, $2bx = aa - cc - bb$; and dividing by $2b$, $x = \frac{aa - cc - bb.}{2b}$

The Arithmetical Rule. Substract from the square of the greater side the sum of the squares of the base and lesser side; and the Remainder divided by double the base will give its continuation to the Perpendicular.

Geometrical Construction from the Equation reduc'd: Having described a semi-circle upon AB (*n.* 2.) apply therein AC, and the $\square$ of CB drawn will $= aa - cc - bb$; which since it must be divided by $2b$ make, as CF $= 2b$ to CE $\sqrt{aa - cc - bb}$, so CE to CD the segment sought, *n* 1.

PROBLEM X.

Commonly ascribed to Archimedes.

THE *Diameter AB of a given semi-circle* (*Fig.* 22 *n.* 1.) *being any how divided in L, and from L erecting a Perpendicular LX, and upon the segments LA and LB having described two other semi-circles, whose semi-diameters are also given as well as that of the great Circle CB; to find the Radii FM and Vy of the little Circles that are to be so described, that they shall touch the Perpendicular LX, the Cavity of the greater semi-circle, and the Convexities of the less.*

SOLV-

SOLVTION.

I. *For the Radius FM.*

1. *Denomination.* Make CB $= a$, EB $= b$ then will CE $= a - b$; for which for brevities ſake put c. And let FM or FN or FK $= x$: Therefore EF will be $= b + x$, and CF (ſubſtracting FK from CK) $= a - x$. Wherefore now you'l have at leaſt the names of the three ſides in the △ CFE, ſo that according to Poblem 8. the Segment of the baſe GE may be determined (which indeed is determined already, as being $=$ LE — LG or MF *i. e.* $b - x$) for which in the mean time we will put y; and now will CG $= c - y$.

2. For the *Equation.* If the □ GE $= y$ be ſubſtracted from the □ EF $= bb + 2bx + xx$, you'l have the ſquare of the Perpendicular FG $= bb + 2bx + xx - yy$; and, if □ CG $= cc - 2cy + yy$ be ſubſtracted from the □ CF $= aa - 2ax + xx$, you'l have the ſame □ of the Perpendicular FG $= aa - 2ax + xx - cc + 2cy - yy$. Therefore $bb + 2bx + xx - yy = aa - 2ax + xx - cc + 2cy - yy$.

3. *Reduction.* And taking from both ſides the quantities xx and yy, $bb + 2bx = aa - 2ax - cc + 2cy$; and adding $2ax$ and cc, but taking away aa from both ſides, $bb + 2bx = aa - 2ax - cc + 2cy$; and adding $2ax$ and cc, and taking away from each ſide aa, $bb + 2bx + 2ax + cc - aa = 2cy$; and dividing by $2c$,

$$\frac{bb + 2bx + 2ax + cc - aa}{2c} = y.$$

but the ſame y or EG is $=$ EL — MF *i. e.* $b - x$. Therefore $\frac{bb + 2bx + 2ax + cc - aa}{2c} = b - x$;

which is a new and more principal Equation: And multiplying both ſides by $2c$ (you have a new Reduction) $bb + 2bx + 2ax + cc - aa = 2ac - 2bx$; and adding $2cx$, and tranſpoſing the others, $2bx + 2cx + 2ax = aa - bb - cc + 2bc$; and dividing by $2a + 2b + 2c$,

$$x = \frac{aa - bb - cc + 2bc}{2a + 2b + 2c}.$$

The

The Geometrical Construction of this first Case. Add the determinate (n 2.) quantity $2bc$ into one ſum with the quantity aa, as in the beginning, n. 3. Then from this ſum ſubſtract ſucceſſively the quantities bb and cc, and there will come out (the ſame n 3) FH, whoſe □ = $aa \dagger 2bc — bb — cc$: Which ſince it muſt be divided by $2a \dagger 2b \dagger 2c$, make (the ſame n. 3) as FI = $2a \dagger 2b \dagger 2c$ to FH = $\sqrt{aa \dagger 2bc — bb — cc}$, ſo FH to FM the Radius ſought of the little Circle to be deſcribed. This quantity FM being thus found, place it from L to G (n. 1.) and from G erect a Perpendicular, which being cut off at the interval CF (which may be had, if from CB or CK you cut off FK = FM) or from E at the interval EF (which is compoſed of the Radii EN and FN) gives the Center of the little Circle to be deſcribed.

The Arithmetical Rule. Add twice the □ CEB to the ſquare of the greateſt ſemi-diameter CB, and from the ſum ſubſtract the Aggregate of the □ □ CE and EB; divide the remainder by the ſum of all the three Diameters, (AB, AL and LB) *i. e.* by double the greateſt AB; and you'l have the Radius FM, *&c.* For Example ſake let a be = 12, b = 4; c will be = 8, and x will be produced = $2\frac{2}{3}$.

II. For the Radius Vy *by help of the obtuſe-angled* △ *DVC.*

1. *Denomination.* CA = a as above, DA or DL = b, and putting x again for the ſought Vy or VK, CV will be = $a — x$, DL or DR = b, and conſequently DV = $b \dagger x$, and DC = $a — b$, for which for brevity's ſake we will put c. Now you'l have at leaſt in Denomination in the △ CVD the three ſides, ſo that according to Problem 9. the ſegment CW may be determined, for which in the mean while we will put y; then will DW = $c \dagger y$, which is the ſame as DL—WL or Vy *i. e.* $b — x$.

2. For the *Equation.* If the □ CW = yy ſubſtract it from the □ CV = $aa — 2ax \dagger xx$, and you'l have the □ of the Perpendicular VW, $aa — 2ax \dagger xx — yy$; and if the □ DW = $cc \dagger 2cy \dagger yy$, ſubſtract it from the □ DV = $bb \dagger 2bx \dagger xx$, and you'l have the ſame □ of the Perpendicular VW = $bb \dagger 2bx \dagger xx — cc — 2cy — yy$ Therefore

$$aa — 2ax \dagger xx — yy = bb \dagger 2bx \dagger xx — cc — 2cy — yy.$$

3. *Redu-*

3. *Reduction.* Therefore taking from both sides xx and yy, $aa - 2ax = bb + 2bx - cc - 2cy$; and adding $2cy$ and $2ax$, $aa + 2cy = bb + 2ax + 2bx - cc$; and substracting aa, $2cy = bb - aa + 2ax + 2bx - cc$, and dividing by $2c$,

$$y = \frac{bb - aa + 2ax + 2bx - cc}{2c}$$

But if you add to the same y or CW DC $= c$, you'l have

$$DW = \frac{bb - aa + 2ax + 2bx - cc + c}{2c} \quad i.\ e.$$

reducing this c to the same Denomination,

$$\frac{bb - aa + 2ax \quad 2bx + cc}{2c} = DW.$$ But the same DW

$= DL - WL = b - x$. Therefore

$$\frac{bb - aa + 2ax + 2bx + cc}{2c} = b - x,$$ and multiplying

$2c$, $bb - aa + 2ax + 2bx + cc = 2bc + 2cx$, and adding $2cx$, and transposing the rest, $2ax + 2bx + 2cx = aa - bb - cc + 2bc$, and dividing by $2a + 2b + 2c$,

$$x = \underline{aa - bb - cc + 2bc}$$ just as above in the first Case.

4. *The Geometrical Construction* therefore will be the same as there. See *Fig.* 23. *n.* 4. and 5.

5. *The Arithmetical Rule* is also the same, but the given quantities in this Example, which the figure of the Problem will shew, thus vary, while a remains 12, b will be 8, and c 4, from which *data* (or given quantities) there will notwithstanding come out again, for x or the Radius Vy $2\frac{2}{3}$.

II. *Some*

II. *Some Examples of simple or pure Quadratick Equations.*

PROBLEM I.

TO *make a Square equal to a given Rectangle*; i. e. *having given the sides of the Rectangle, to find the side of an equal Square*, Eucl. Prop. 14. Lib. 2. Suppose *e. g.* the given sides of the Oblong to be AB and BC (*Fig.* 24) to find the Line BD whose square shall be equal to that Rectangle.

SOLUTION.

Make AB $=$ a and BC $=$ b, and the side of the square sought $=$ x, and the Equation will be $ab = xx$; and extracting the root on both sides $\sqrt{ab} = x$.

Geometrical Construction. Join AB and BC in one right line, and describing a semi-circle upon the whole AC, from the common juncture B erect the Perpendicular BD which will be the side of the square sought, according to Case 1. of the Effection of pure quadraticks.

Arithmetical Rule. Multiply the given sides of the Oblong by one another, and the square root extracted out of the Product will be the side of the square sought.

PROBLEM II.

THE *square of the Hypothenusa in a right-angled* △ *being given, as also the difference of the other two squares to find the sides.* E. g. If the Hypothenusa be BC (*Fig.* 25.) and the difference of the squares of both the legs, and consequently its Leg also BE given (for the squares being given the sides are also given geometrically) to find the sides of the right-angled △ which shall have these conditions; or more plainly, to find one side *e. g.* the lesser which being found, the other, or the greater, will be found also.

SOLU-

SOLVTION.

Let the □ of the given Hypothenuſa $= aa$, and the ſquare by which the two other differ $= bb$ Let the leſs ſide $= x$, and its □ $= xx$. Wherefore the greater will be $xx + bb$. And ſince the ſum of theſe is = to the □ of the Hypothenuſa, you'l have

$$2xx + bb = aa; \text{ and ſubſtracting } bb,$$
$$2xx = aa - bb; \text{ and dividing by } 2$$
$$xx = \frac{aa - bb}{2}. \text{ Therefore}$$
$$x = \sqrt{\frac{aa - bb}{2}}.$$

Geometrical Conſtruction. Having deſcribed a ſemi-circle on BC, and applyed therein BE, the □ EC will $= aa - bb$; and having deſcribed another ſemi circle upon EC divided into two Quadrants the □ DC will be $\frac{aa-bb}{2}$, and ſo DC $= \sqrt{\frac{aa-bb}{2}}$, or the ſide ſought; which being alſo transferr'd upon the other ſemi-circle deſcrib'd on BC, *viz.* from C to A gives the other ſide AB and the whole △ ſought.

The Arithmetical Rule. From the ſquare of the Hypothenuſa ſubſtract the given difference, and the ſquare root extracted out of half the remainder gives the leſſer ſide of the △ ſought.

PROBLEM III.

H*Aving an equilateral* △ *ABC given* (*Fig.* 26. *n* 1.) *to find the Center and Semi diameter of a Circle that ſhall circumſcribe it.* i. e. Find the BD the ſide of an Hexagon that may be inſcribed in it. For if we conſider the thing as already done; it will be manifeſt that the ſide of the Hexagon BD will fall perpendicularly on the ſide of the △ AB, as making an angle in a ſemi circle, ſo having biſected the Hypothenuſa DA you'l have E the Center ſought.

SOLUTION.

Make the ſide of the Triangle AB $= a$, BD $= x$, then will AD $= 2x$. Since therefore the ſquare BD *i. e.* xx being ſubſtracted out of the ſquare AD *i. e.* $4xx$, there remains the ſquare AB $3xx$, you'l have the Equation

$$3xx = aa\text{; and dividing by } 3$$

$$xx = \frac{aa}{3}\text{; therefore}$$

$$x = \sqrt{\frac{aa}{3}}.$$

The Geometrical Conſtruction. Having produced AB ($n. 2.$) to F a third part of it, the ſquare of a mean proportional BD between BF and BA will be $\frac{1}{3}aa$ or $\frac{aa}{3}$, and ſo the Line BD $= \sqrt{\frac{aa}{3}}$. Therefore the Hypothenuſa DA being divided in two in E, or at the interval BD, making the interſection from B and A, you'l have the Centre ſought.

The Arithmetical Rule. Divide the ſquare of the given ſide into three equal parts, and the ſquare Root of a third part will give the ſemi-diameter AE or BE ſought, by the interſection of two of which you'l have the Centre.

PROBLEM IV.

Having given, in a right-angled Parallelogram, the Diagonal, or for a right-angled △, *the Hypothenuſa and the proportion of the ſides, to find the ſides ſeparately and conſtruct the Parallelogram or* △. Suppoſe *e. g.* the given Diagonal to be AB (*Fig.* 27. *n* 1.) and the given reaſon of the ſides as AD to DE, to find the ſides.

SOLUTION.

Make AB $= a$, the reaſon of AD to DE as b to c; make the leſſer ſide $= x$, then will the greater be $\frac{cx}{b}$.

For

For the *Equation*, the □□ of the sides are $xx + \frac{ccxx}{bb} = aa$, □ AB; and multiplying both sides by bb,

$$bbxx + ccxx = aabb;$$ and dividing by $bb + cc$,

$$xx = \frac{aabb}{bb+cc}.$$ Therefore

$$x = \sqrt{\frac{aabb}{bb+cc}}$$ i. e. Extracting the roots as far as possible

$$x = \frac{ab}{\sqrt{bb+cc}}$$

Another Solution.

Call the name of the given reason e, so that assuming any line for unity, the value of e may also be expressed by a right line, which shall be equal *e. g.* to DE above. Wherefore because we make the less side x, the greater will be ex, and so

$$xx + eexx = aa,$$ i. e. dividing by $1 + ee$,

$$xx = \frac{aa}{1+ee}$$

$$x = \sqrt{\frac{aa}{1+ee}},$$ i. e.

$$x = \frac{a}{\sqrt{1+ee}}$$

The Geometrical Construction. The last Equation above being reduc'd to this proportion as the $\sqrt{bb+cc}$ to b so a to x, make (n 2) AD and DE at right angles, and AE will be $= \sqrt{bb+cc}$, and continuing AE and AD make as AE to AD so AB to AC the lesser side sought. Having therefore drawn BC which determines the lesser side AC, the greater side and so the △ ABC will be already formed, and may be easily compleated into a Rectangle. In the other Solution the last Equation agrees with the precedent (for it gives us this proportion as $\sqrt{1+ee}$ to 1 so a to x in which 1 is $= b$, and $ee = cc$ by what we have supposed) and so the Construction will be the same.

The Arithmetical Rule may be more commodiously expreſſed by this laſt Equation under the laſt form but one, after this way, divide the □ of the Diagonal by the □ of the name of the Reaſon leſſen'd by unity, and the root extracted out of the Remainder is the leſſer ſide ſought.

PROBLEM V.

(Which is in *Pappus Alexandrinus*, and in *Cartes's* Geometry, p. 83. in a Biquadratick affected Equation, and p. 84. he gives us thereon a very remarkable Note.)

H*Aving given the Square AD (Fig. 28) and a right line BN, you are to produce the ſide AC to E, ſo that EF drawn from E towards B ſhall be equal to BN.*

It will be evident, if you imagine a ſemi-circle to paſs thro' the points B and E, that the moſt commodious way will be to find the line DG, that you may have the Diameter BG; upon which having afterwards deſcribed a ſemi circle, there will be need of no other operation to ſatisfie the queſtion, than to produce the ſide AC 'till it occur to the preſcrib'd Periphery.

SOLUTION.

(*As found by* Van Schooten, p. 316. *in his Comment on* Cartes's *Geometry, which we will here give ſomewhat more diſtinct.*)

1. *Denomination.* Make BD or DC = a, BN or FE = c, BF = y, and DG = x; the Perpendicular EH will be = a, and EG = BF, *viz* y (becauſe the △ EHG is ſimilar to △ FDF, by *n.* 3 *Schol* 2 *Prop* 34. *Lib.* 1. *Matheſ. Enucl.* and BD in the one = to EH in the other) and BG = a † x, BE = y † c; and BH will have its Denomination, if you make (by reaſon of the ſimilarity of the △ △ BFD and BEH)

as BF to BD ſo BE to BH

y —— a —— $y + c$ —— $\frac{ay + ac.}{y}$

and you'l have alſo HG $= a + x - \frac{ay - ac}{y}$, i. e. having reduc'd them all to the ſame Denomination,

$\frac{ay}{y} + \frac{xy}{y} - \frac{ay}{y} - \frac{ac}{y}$, i. e. $\frac{xy - ac}{y}$. Having therefore named all the lines you have occaſion for, you muſt find two Equations, becauſe there are aſſumed two unknown quantities, *viz.* x and y.

2. *For the firſt Equation and its Reduction.* By reaſon of the ſimiliarity of the △ △ BGE and BEH,

as BG to GE ſo BE to EH

$a + x$ —— y —— $y + c$ —— a: Therefore the Rectangle of the Extremes will be $=$ to the Rectangle of the means, *i. e.* $yy + yc = aa + ax$; and taking from both ſides yc,

$yy = aa + vx - yc.$

3. *For the ſecond Equation and its Reduction.* Since BH, HE and HG

$\frac{ay + ac}{y}$ — a —— $\frac{xy - ac}{y}$ are continual proportionals, the Rectangle of the extreams are equal to the ſquare of the mean, *i. e.* $\frac{axyy + acxy - aacy - aacc}{yy} = aa$; and multiplying both ſides by yy, and dividing by a

$xyy + cxy - acy - acc = ayy$, and taking away ayy and tranſpoſing the reſt,

$xyy - ayy = acy - cxy + acc$; and dividing by $x - a$

$yy = \frac{acy - cxy + acc}{x - a}$; i. e. dividing actually as far as may be by $x - a$, $yy = - cy + \frac{acc.}{x - a}$

4. The compariſon of theſe two Equations thus reduc'd, gives a third new one, in which there will be only one unknown quantity, *viz.*

$- cy + aa + ax = - cy + \frac{acy}{x - a}$; and adding to both ſides cy,

aa

$aa + ax = \frac{acc}{x-a}$; and multiplying by $x-a$,

$aax + axx - a^3 - aax = aac$; i. e. $axx - a^3 = ac$
and dividing both ſides by a, $xx - aa = cc$; and adding
$xx = aa + cc$. Therefore $x = \sqrt{aa + cc}$.

5. *The Geometrical Conſtruction*, which is the ſame Pap
preſcribes in *Cartes*, *viz.* having prolong'd the ſide of th
ſquare BA to N, ſo that BN ſhall be = to a given right line,
ſince BA is $= a$, and BN $= c$, the Hypothenuſa DN will b
$= \sqrt{aa + cc} = x$. Having therefore made DG = DN, and
deſcrib'd a ſemi-circle upon the whole line BG, if AC be pro-
longed until it occur to the Periphery in E, you'l have done
that which was requir'd.

PROBLEM VI.

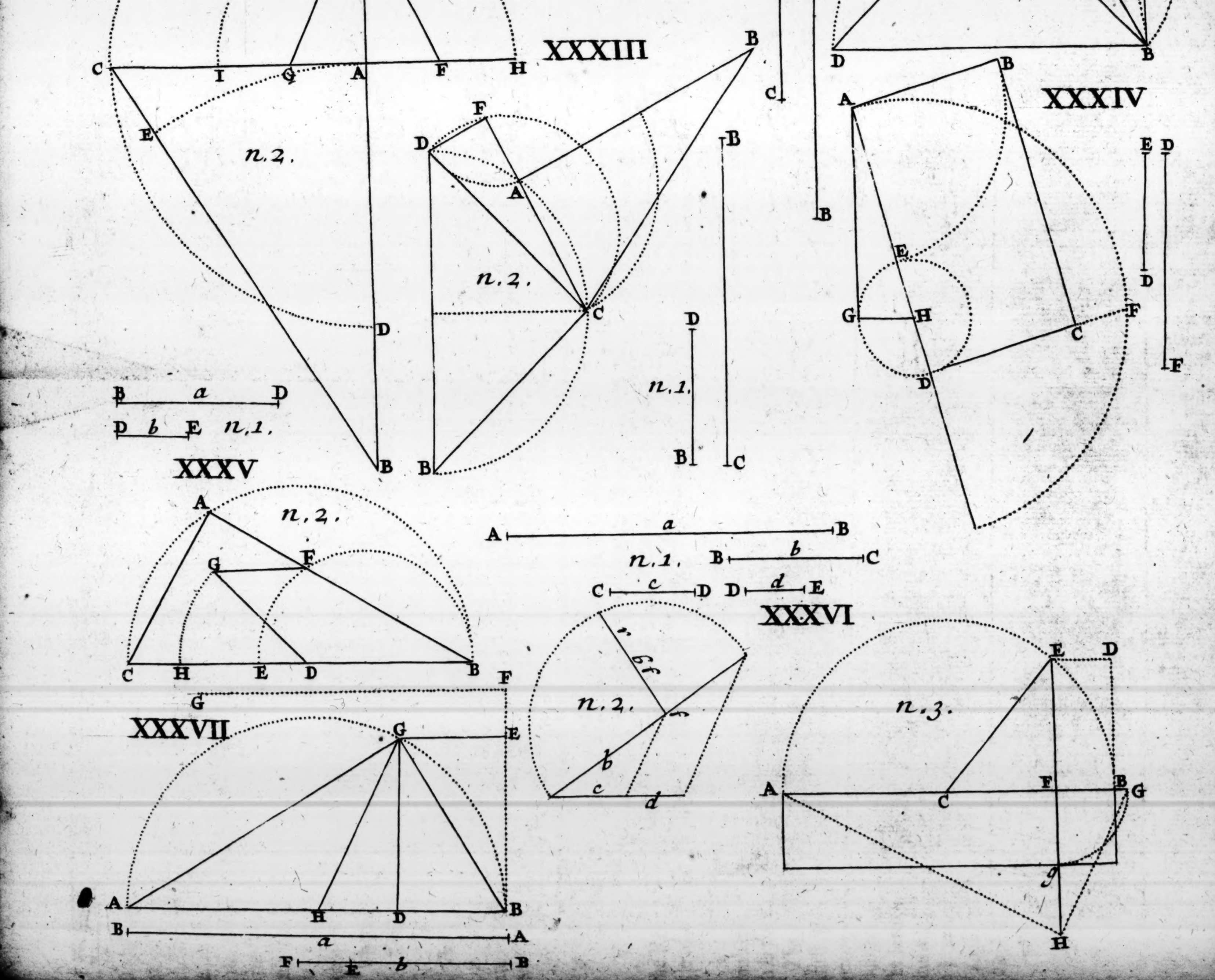
(Which *Van Schooten* has in his *Comment*,
p. m. 150, and following.)

H*Aving given a right line AB, from the ends of it A and B (Fig. 29.) to inflect two right lines AC and BC, which ſhall contain an angle ACB = to the given one D, and whoſe ſquares ſhall be in a given proportion to the Triangle ACB,* viz. *as 4d to a.*

Viz. You muſt determine the point C, which the two right lines AH and HC or EH and HC will do, aſſuming the mid-dle point E in the line AB. Wherefore here will be two un-known quantities HE and HC, and conſequently two Equa-tions to be found in the Solution; one whereof the given proportion in the Queſtion ſupplies us with, and the other we have from the ſimilar Triangles AIC and GFD, which repre-ſent equal angles.

SOLUTION.

Denomination. Make AE, half AB $= a$, HE $= x$ and HC $= y$; therefore AH will be $= a - x$ and HB $= a + x$; whence the Denomination of the ſquares AC and BC is eaſily had; *viz.* the one $aa - 2ax + xx + yy$, and the other, $aa +$
$2ax$


XXXIII
n.2.
n.1.
XXXIV
XXXV
n.2.
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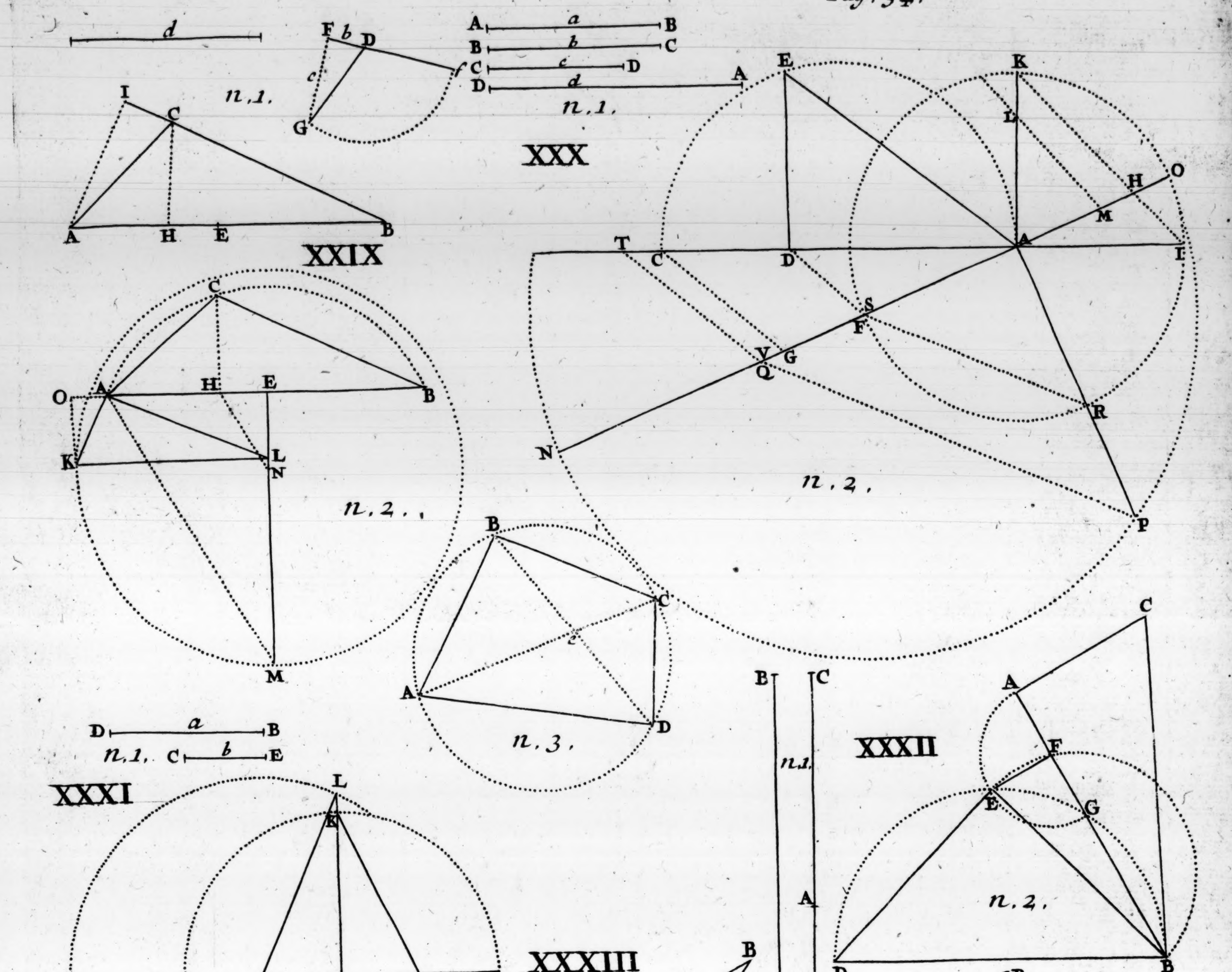
XXXVI
n.3.
XXXVII


XXIX
n,1.
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XXX
n,1.
n,2.
n,3.
XXXI
n,1.
XXXII
n,1.
n,2.
XXXIII

$+xx+yy$, so that the sum of the squares is $2aa+2xx+$
And the $\triangle$ ACB will be $= ay$: And since the $\triangle\,\triangle$
D and AIC are similar, and the sides of the former FD and
arbitrary, so that for FD we may put b and for FC, c;
the sides of the latter are determined by the similitude of
$\triangle\,\triangle$ ABI and HDB, as being right-angled ones, and ha-
the common angle B; they will be obtain'd by making
the Hypothen. BC to the Hypoth. AB so the base HB to
base BI, *i. e.*

$$e \text{ — } 2a \text{ — } a+x \text{ — } \frac{2aa+2ax}{e},$$

whence substracting BC $= e$, there remains CI $=$
$$\frac{2aa+2ax-ee}{e}.$$

For the first Equation, by virtue of the Problem as $4d$
so $2aa+2xx+2yy$ to ay. And the Rectangle of the
mes is $=$ to the Rectangle of the means, *i. e.* $4ady = 2a^3$
$xx+2ay$.

For the other Equation, since
as DF to FG so CI to AI

$$b \text{ — } c \text{ — } \frac{2aa+2ax-ee}{e} \text{ — } \frac{2ay}{e}$$

the Rectangle of the extreams will again be $=$ to the Re-
gle of the means, i. e. $\frac{2ayb}{e} = \frac{2aac+2acx-cee}{e}$;
multiplying both sides by e, $2ayb = 2aac+2acx-cee$;
h is the second Equation.

The Reduction of both Equations.
first was $4ady = a^3+2axx+2ayy$.
refore dividing by $2a$, $2dy = aa+xx+yy$.
substracting $aa+yy$, $2dy-aa-yy = xx$.
The latter Equation was $2ayb = 2aac+2acx-eec$, i. e.
tituting again the value ee, which was
$a+2ax+xx+yy$.

$2ayb = 2aac+2acx-aac-2acx-cxx-cyy$,
$2ayb = aac-cxx-cyy$; and by transposition,
$cxx = aac-2ayb-cyy$; and dividing by c,
$xx = \frac{aa-2ay-yy}{c}$, or $aa-yy-\frac{2ayb}{c}$,

or

or (putting $2f$ for $\frac{2ab}{c}$) $xx = aa - yy - 2fy$.

Wherefore we have the value of xx twice expressed, but by quantities partly unknown, because y is found on both sides. Wherefore now we must make a new comparison of their values, whence you'l have this new

5. *Third Equation*, in which there is only one of the unknown quantities:

$2dy - aa - yy = aa - yy - 2fy$; and adding on both sides both yy and aa,

$2dy = 2aa - 2fy$; or dividing by 2,

$dy = aa - fy$; and transposing fy,

$dy + fy = aa$; and dividing by $d + f$,

$y = \frac{aa}{d+f}$; which is the value of the quantity y in known terms.

But this value in one of the precedent Equations, *viz.* in this $xx = 2dy - aa - yy$, being substituted for y and its square for yy, will give

$$xx = \frac{2daa}{d+f} - aa - \frac{a^4}{dd+2df+ff};$$ i. e. all being reduced to the same denomination,

$$xx = \frac{aadd - aaff - a^4}{dd+2df+ff};$$ and

$$x = \sqrt{\frac{aadd - aaff - a^4}{dd+2df+ff}}.$$

6. *The Geometrical Construction*, which *Schooten* gives us p. 153. Having made the angle KAB (*n.* 2. *Fig.* 29.) equal to the given one D, erect from A, AL perpendicular to KA, meeting the Perpendicular EM in L; and from the Centre L, at the interval of the given right line *d*, describe a Circle that shall cut KA and EL produced to K and M. Then assuming EN = KA, join MA, and from N draw NH parallel to it, which shall meet AB in H. Afterwards, having described from L, at the interval LA, the segment of a Circle ACB, draw from H, HC perpendicular to AB meeting the circumference in C, and join AC, CB.

NB. The

NB. The reason of this elegant Construction, which the Author conceal'd, for the sake of Learners we will here shew: 1. Therefore, he reduc'd the last Equation (extracting the root, as well as it could bear, both of Numerator and *Denominator*) to this: $x = \frac{a}{d+f}$ multipl. by $\sqrt{dd - ff - aa}$, so that after this way the Construction would be reduc'd to this proportion, as $d+f$ to a so $\sqrt{dd-ff-aa}$ to x. 2. He made the angle KAE = to the given one D, and the angle KAL a right one, so that having described the segment of a Circle from L the inscribed angle will also be made equal to the given one, according to the 33. *Lib.* 3. *Eucl.* 3. By doing this, EL expresses the quantity f, since by reason of the similarity of the △ △ KOA f. GFD, *n.* 1. and AEL (for the angles LAE and AKO are equal, because each makes a right one with the same third Angle KAO) you have

as KO to OA so AE to EL i. e. f

c —— b —— a —— $\frac{ab}{c}$

4. Making now LM and LK = d you had EM = $d+f$, and AK = $\sqrt{dd-ff-aa}$ (for the □ AL is = to $aa+ff$, which being substracted from □ LK = dd, there remains □ AK = $dd - aa - ff$.)

5. Wherefore there now remains nothing to construct the last Equation above, but to make EN = AK, and to draw HN parallel to AM; for thus was the whole proportion

as EM to EA so EN to EH

$d+f$ —— a —— $\sqrt{dd-ff-aa}$ to x. Q. e. f.

For the point H being determined, a perpendicular HC thence erected in the segment already described defines the Point C, which answers the Question.

PROBLEM VII.

Having given the four sides of a Quadrangle to be inscribed in a Circle, to find the Diagonals and their Segments, and so to construct the Quadrangle, and inscribe it in the Circle. As e. g. suppose the given sides are AB, BC, CD, DA (*Fig.* 30 *n.* 1.) which now we suppose to be joined into a quadrangle

gle inſcrib'd in the Circle, the Diagonals alſo AC and BD being drawn (*n.* 3.) to find firſt the ſegments of the diagonals A*e*, B*e*, &c. which being had, the Conſtruction is ready.

SOLUTION.

Denomination. Make AB $= a$, AC $= b$, CD $= c$, DA $= d$, A*e* $= x$ [for this ſegment alone being found, the reſt will be found alſo, as will be evident from the proceſs.] Since therefore the vertical angles at *e* are equal, and likewiſe the angles in the ſame ſegment BCA, BDA, alſo DAC, DBC, &c. are equal, the Triangles A*e*D and B*e*C, alſo A*e*B and C*e*D are ſimilar: wherefore it will follow that,

1. As DA to A*e* ſo CB to B*e*
$$d \text{ --- } x \text{ --- } b \text{ --- } \frac{bx}{d}$$
2. As AB to B*e* ſo CD to C*e*
$$a \text{ --- } \frac{bx}{d} \text{ --- } c \text{ --- } \frac{bcx}{ad}$$
3. As AB to A*e* ſo CD to D*e*
$$a \text{ --- } x \text{ --- } c \text{ --- } \frac{cx}{a}$$

Therefore the whole Diagonal AC will be $= x + \frac{bcx}{ad}$ and BD $= \frac{bx}{d} + \frac{cx}{a}$.

2. *The Equation.* But now by *Prop.* 48. *Lib.* 1. *Math. Enucl.* the Rectangle of the Diagonals is equal to the two Rectangles of the oppoſite ſides.

$$\text{Diag. AC, } x + \frac{bcx}{ad}$$
$$\text{Diag. BD, } \frac{bx}{d} + \frac{cx}{a}$$
$$\overline{\qquad\qquad\qquad\qquad}$$
$$\frac{cxx}{a} + \frac{bbcxx}{add}$$

Therefore

Therefore

the $\square$ of the Diagonals $\frac{bxx}{d} + \frac{cxx}{a} + \frac{bbcxx}{add} + \frac{bccxx}{aad}$ is

$= ac + bd.$

3. *Reduction*, i. e. taking (*a*) for unity

$$\frac{bxx}{d} + cxx + \frac{bbcxx}{dd} + \frac{bccxx}{d} = c + bd;$$

i.e. the quantities on the left hand being reduc'd to the same denomination.

$$\frac{bdxx + cddxx + bbcxx + bccdxx}{dd} = c + bd;$$

and multiplying both sides by dd,

$$bdxx + cdxx + bbcxx + bccdxx = cdd + bd^3;$$

and dividing both sides by d

$$bxx + cxx + \frac{bbcxx}{d} + bccxx = cd + bdd;$$

and then dividing both sides by $b + cd + \frac{bbc}{d} + bcc$,

$$xx = \frac{cd + bdd}{b + cd + \frac{bbc}{d} + bcc};$$ i. e. in the present case, where b by chance happens to be $= a$,

$$xx = \frac{cd + dd}{1 + cd + \frac{c}{d} + cc}$$

Therefore $x = \sqrt{\frac{cd + bdd}{b + cd + \frac{bbc}{d} + bcc}}$

or in our case

$$x = \sqrt{\frac{cd + dd}{1 + cd + \frac{c}{d} + cc}}$$

4. *The Geometrical Construction*, which, by supposing a (and in the present case also b) to be unity, ought to deter-

mine, 1. The quantities cd, $\frac{c}{d}$, cc, and their aggregate with unity. 2. The aggregate of cd and dd. 3. To divide the one by the other. And, 4. To extract the root out of the quotient, or also to extract the roots first out of each quantity, and divide them by one another; which may all of them be separately done in so many separate Diagrams, but more elegantly connected together after the following or some such like way. 1. Join AD and DC (*n.* 2.) into one line, and having described a semi-circle thereupon, erect the Perpendicular DE; and the line AE drawn will $= \sqrt{cd + dd}$. 2. Making the angle CAG at pleasure, make AF $=$ AB, and draw CG parallel to the line DF; so FG will be $= \frac{c}{d}$. Now if, 3. in the vertical angle you make AH $=$ CD, the line HI drawn parallel to DF will cut off AI $= cd$. 4. In AK erected $\perp$ to AB, if you take AL $=$ AH or CD, and draw LM parallel to KH, you'l have AM $= cc$. 5. Having prolonged AG to N and AH to O, so that GN shall be $=$ AI $+$ AM and AO $=$ AB or AK, and having described a semi-circle upon the whole line NO, a perpendicular erected AP will be $= \sqrt{1 + cd + \frac{c}{d} + cc}$; and so, 6. if AQ be made $=$ AE and AR $=$ AF or AB, and you draw a line RS from R parallel to PQ; AS will be $= x$, *i. e.* the segment sought A*e* of the Diagonal AC; which being given, by force of the first Inference premis'd in the Denomination above, by drawing DS and, having made DT $=$ BC, TV parallel to it; you'l have also the other segment B*e* $=$ SV and by their Intersection on the line AB (*n.* 3.) the point *e*, thro' which the Diagonals must be drawn which will be terminated by the other given sides, and thence you'l have the quadrilateral figure ABCD sought, to be circumscribed about the Circle, according to *Consect.* 6 *Defin.* 8. *Mathes. Enucl.*

NB. Unless we had here consulted the Learner's ease, the artifice of this Construction might be proposed after a more short and occult way, thus: Make DE a mean proportional between AD and DC, and draw AE. Then having made any angle CAG, make AF $=$ AB, and at this Interval describe

scribe the circle FROK, and draw CG parallel to DF. Moreover in the opposite vertical angle, having made AH = CD, draw HI parallel to DF, and having erected the perpendicular AK, and thence the abscissa AL = AH, make LM parallel to HK, and thence having prolonged AH to O, and GN being made equal to AI + AM, make AP a meam proportional between AO and AN, cutting the hidden circle in R; and lastly having made AQ = AE, if RS be drawn parallel to QP, you'l have AS the value of x sought, *&c.*

III. *Some Examples of Affected Quadratick Equations.*

PROBLEM I.

H*Aving given, to make a right angled Triangle ABC, the differences of the lesser and greater side, and of the greater, and the Hypothenusa, to find the sides separately and form the Triangle.* E. g. Having given the right line DB (*Fig.* 31.) for the difference of the perpendicular and base, and CE for the difference of the base and Hypothenusa, to find the perpendicular AC, which being found, you'l have also, by what we have supposed, the base AB, and the hypothenusa BC.

SOLUTION.

Make the difference DB $= a$, CE $= b$; put x for the perpendicular; the base, which is greater than that will be $x + a$ and the Hypothenusa $x + a + b$. Therefore by vertue of the Pythagorick Theorem,

$$2xx + 2ax + aa = xx + 2ax + 2bx + aa + 2ab + bb;$$

and substracting from both sides $xx + 2ax + aa$,

$$xx = 2bx + 2ab + bb.$$

Wherefore by the first case of affected quadratick Equations

$$x = b + \sqrt{2ab + 2bb}.$$

Construction. Find a mean proportional AK between AH $= 2b$ and AI $= a$ (*n.* 2.) (*Fig.* 31.) and having made both AF and AG $= b$, place the Hypothenusa KF from AL, and cut

cut off GC equal to the hypothenuſa GL; thus you'l have AC the perpendicular of the Triangle ſought, and adding DB you'l alſo have the baſe AB, and from thence having drawn the hypothenuſa BC, it will be found to differ by the exceſs required CE.

The Arithmetical Rule. Join twice the product of the differences multiplyed by one another, to twice the ſquare of the difference of the baſe and the hypothenuſa; and if the ſquare root of this ſum being extracted be added to the aforeſaid difference, you'l have the perpendicular ſought. Suppoſe *e.g.* both the differences of CE and DB = 10.

PROBLEM II.

IN *a right-angled △ having given the Hypothenuſa and ſum of the ſides, to find the ſides.* E.g. If the Hypothenuſa BC be given (*Fig.* 32.) and the ſum of the ſides CAB, to find the ſides AB and AC ſeparately, to form the Triangle.

SOLUTION.

Make the Hypothenuſa BC = a, the ſum of the ſides = b. Make one ſide *e.g.* AB = x, then will the other ſide AC be = $b - x$. Therefore

$2xx - 2bx + bb = aa$; and adding $2bx$, and taking away bb, $2xx = 2bx + aa - bb$; and dividing by 2,

$$xx = bx + \frac{aa - bb}{2}.$$

Therefore according to Caſe 1. of affected Quadraticks,

$$x = \tfrac{1}{2}b + \sqrt{\tfrac{1}{4}bb + \frac{aa - bb}{2}}$$

$$\text{i. e. } \tfrac{1}{2}b + \sqrt{\tfrac{1}{2}aa - \tfrac{1}{4}bb}$$

$$\text{or } \tfrac{1}{2}b - \sqrt{\tfrac{1}{2}aa - \tfrac{1}{4}bb}.$$

The Geometrical Conſtruction. Having deſcribed a ſemi-circle upon BD = BC ſo a apply therein the equal lines BE and DE, and having deſcribed another ſemi-circle on BE apply therein BF = $\tfrac{1}{2}b$, to be prolonged farther out. Laſtly, if another little ſemi circle be deſcribed at the interval EF, the whole line AB will

will be the true root or the side sought, and GB the false root, &c.

The Arithmetical Rule. From half the square of the Hypothenusa substract the fourth part of the square of the given sum, and the root extracted out of the remainder, if it be added to half the sum, will give one side of the Triangle; and substracted from the given sum, will give also the other Suppose *e. g.* BC to be 20, and the sum of the sides 28.

PROBLEM III.

Having given again in the same △ the Hypothenusa, as above, and the difference of the sides DB (Fig. 33.) to find the sides.

SOLUTION.

Make the less side x, the difference of the sides $= b$; the greater side will be $x + b$. Let the Hypothenusa be $= a$. Therefore, $2xx + 2bx + bb = aa$; and taking away $2bx + bb$,

$$2xx = aa - 2bx - bb; \text{ and dividing by } 2,$$

$$xx = -bx + \frac{aa - bb}{2}.$$

Therefore by case 2, $x = -\frac{1}{2}b + \sqrt{\frac{1}{4}bb + \frac{aa - bb}{2}}$

$$\text{i. e. } -\tfrac{1}{2}b + \sqrt{\tfrac{1}{2}aa - \tfrac{1}{4}bb}.$$

The Geometrical Construction. Having described a semi-circle upon BD $=$ BC or a, apply therein the equal lines BC and DC, and having described another semi circle on DC apply in it DF $= \frac{1}{2}b$, and if at the same interval you cut off FA from FC, the remainder AC will be the lesser side sought, &c.

The Arithmetical Rule. From half the square of the Hypothenusa substract the square of half the difference, and if you take half the difference from the root extracted out of the remainder, you'l have the lesser side of the Triangle required, and by adding to it the given difference you'l have also the greater. *E. g.* Let the Hypothenusa be 20, and the difference of the sides 4.

PRO-

PROBLEM IV.

Having given the Area of a right-angled Parallelogram, and the difference of the sides to find the sides. E. g. If the Area is = to the square of the given line DF, and the difference of the sides ED (*Fig.* 34.) to find the sides of the rectangle.

SOLUTION.

Make the given Area $= aa$, the difference of the sides $= b$, the lesser side x; then the greater will be $x + b$. Therefore the Area $xx + bx = aa$; and substracting bx

$$xx = -bx + aa.$$

Therefore according to case 2, $x = -\frac{1}{2}b + \sqrt{\frac{1}{4}bb + aa}$.

The Geometrical Construction. Join at right angles AG $= a$, and GH $= \frac{1}{2}b$, and having drawn AH and prolong'd it, describe the little Circle at the interval GH: so you'l have AE the lesser side, and AD the greater of the Rectangle sought, &c.

The Arithmetical Rule. Add the given Area and the square of half the difference, and having the sum, substract and add the difference from or to the root extracted, and so you'l have the greater and less sides of the rectangle.

PROBLEM V.

Having given for a right-angled Triangle the difference of both the Legs from the Hypothenusa, to find the sides and so the whole Triangle. E. g. Suppose the difference of the less side to be BD and of the greater DE (*n.* 1. *Fig.* 35.) to find the sides themselves, and so make the Triangle.

SOLUTION.

For BD put a, for DE, b. Let the greater side be x; the Hypothenusa will be $x + b$; therefore the lesser side will be $x + b - a$. Now the □ □ of the sides are = to the □ of the Hypothenusa, *i. e.* $2xx - 2ax + 2bx + bb - 2ab + aa = xx$

$= xx + 2bx + bb$; and taking away $xx + 2bx + bb$,

$$xx - 2ax - 2ab + aa = 0$$; and adding $2ax$ and $2ab$, and taking away aa,

$$xx = 2ax + 2ab - aa.$$ Therefore

$$x = a + \sqrt{aa + 2ab - aa};$$ i. e.

$$x = a + \sqrt{2ab}.$$

The Geometrical Construction. Between the given differences BD and DE find (*n* 2.) a mean proportional DF, and join to it at right angles the equal line FG, and cut off DH equal to DG; and so you'l have BH the greater side of the triangle sought. This being prolong'd to C, so that HC shall be $= b$, having described a semi-circle upon the whole line BC apply therein BA $=$ BH; and having drawn AC, the Triangle sought ABC, will be formed.

The Arithmetical Rule. If the square root extracted from the double rectangle of the differences be added to the greater difference, you'l have the greater side sought, *&c.*

PROBLEM VI.

Having given, to make two unequal Rectangles, but of equal heighth, the sum of their Bases with the Area of either (viz. *the greater,*) *and the proportion of the sides of the other* (viz *the least,*) *to find the sides separately.* E. g. Let the sum of the bases be AB (*n.* 1. *Fig.* 36) and the square of the line BC $=$ to the Area of the greater rectangle; and let the sides of the lesser rectangle be to one another as CD to DE: To find the sides of both the rectangles; *i. e.* to find the common altitude, which being found the other sides will be easily obtain'd from the *Data*; or to find the base of the greater which, with the same ease, will discover the rest.

SOLUTION.

Make AB $= a$, and the Area of the greater rectangle $= bb$; and the proportion of the altitude to the base in the lesser, as c to d; to find *e. g.* the greater base which call x. Therefore the common altitude will be $= \frac{bb}{x}$, and the base of the lesser rectangle $= a - x$.

 Where-

Wherefore you'l have for the Equation,

$$\text{as } c \text{ to } d \text{ so } \frac{bb}{x} \text{ to } a - x.$$

Therefore $ac - cx = \frac{bbd}{x}$; and multipl. by x,

$acx - cxx = bbd$; and adding cxx and taking away bbd,

$acx - bbd = cxx$. Now that you may conveniently divide both sides by c, make first as c to b so d to a fourth which call f, and then put cf for bd, and you'l have

$acx - bcf = cxx$; and dividing by c

$ax - bf = xx$; and so according to case 3.

$$x = \tfrac{1}{2}a + \sqrt{\tfrac{1}{4}aa - bf}$$

$$\text{or } \tfrac{1}{2}a - \sqrt{\tfrac{1}{4}aa - bf}.$$

The Geometrical Construction. Find first the quantity f (*num.* 2. *Fig.* 36.) according to the following proportion, as c to b so d to f; and a mean proportional between b and f will be $= \sqrt{bf}$. Then having at the interval $\frac{1}{2}a$ described a semi circle (*n.* 3.) upon the given line AB, and erected BD $= \sqrt{bf}$, and having made EF $=$ to it, CF will be $\sqrt{\frac{1}{4}aa - bf}$: To which AC being added will give x for one value and FB for the other. And for the common altitude, which we called $\frac{bb}{x}$, make as x to b, so b to a fourth, *i. e.* as AF to FH so FH to FG; which will be the altitude of both rectangles Ag and Bg which may now easily be constructed.

The Arithmetical Rule might easily be had from this Equation reduced; but you may have it more commodiously from this other

SOLUTION.

Let the Denomination remain the same as above, only here put x for the common altitude, and express the reason of the lesser base of the rectangle to this altitude by e, and that base will be $= ex$: Therefore the base of the greater Rectangle will be $= a - ex$. Having now multiplyed the common altitude by each base, the area of the greater rectangle will be $ax - exx$, and hence you'l have the Equation

$ax - exx = bb$; and adding exx, and taking away bb, $ax - bb = exx$; and dividing by e,
$\frac{ax - bb}{e} = xx$. Therefore by case 3.

$$x = \frac{a}{2e} + \sqrt{\frac{aa}{4ee} - \frac{bb}{e}} \text{ or } x = \frac{a}{2e} - \sqrt{\frac{aa}{4ee} - \frac{bb}{e}}.$$

Wherefore now this will be the *Arithmetical Rule.* If from the fourth part of the square of the sum of the bases divided by the □ of the name of the reason you substract the given area divided by the same name of the reason, and if the root extracted out of the remainder be added to or substracted from half the sum of the bases divided by the same name of the reason; this sum or remainder will give the altitude of the given Rectangles, and that multiplyed by the name of the reason one of the bases: and that being substracted from the given sum of the bases will give the other base. For Example, let the sum of the bases be 16, the area of one of the rectangles 30, the name of the reason which the common altitude has to the base of the other rectangle = 2. There will come out the common altitude, on the one side 5, on the other 3, *&c.*

PROBLEM VII.

Having given the Perpendicular of a right-angled Triangle let fall from the right angle, and its Base, to find the segments of the Base, and so to form the Triangle.

E g. If the base of the right angled Triangle you are to form be AB (*Fig.* 37.) and the length of the perpendicular CD or BF; to find the segments of the base, and so the point D from which you are to make the perpendicular CD, to form the Tiangle ABG.

SOLUTION.

Let the given base be = a; and the given perpendicular = b: Then will one of the segments of the base be = x; and the other = $a - x$, and b a mean proportional between the said segments, *i. e.*

x to b as b to $a - x$;

Therefore $ax - xx = bb$; and by adding xx and taking away bb,

ax—

$ax - bb = xx$. Therefore by case 3.

$$x = \tfrac{1}{2}a + \sqrt{\tfrac{1}{4}aa - bb}$$

$$\text{or } \tfrac{1}{2}a - \sqrt{\tfrac{1}{4}aa - bb}.$$

The Geometrical Construction. Having described a semi-circle upon the given line AB, if you erect the perpendicular BE, and from the point G (which is determined by EG parallel to AB) let fall GD equal to it, you will have the two segments sought, AD $= \frac{1}{2}a + \sqrt{\frac{1}{4}aa - bb}$ and DB $= \frac{1}{2}a - \sqrt{\frac{1}{4}aa - bb}$, which Construction, it cannot be denyed, but it may be evident to any attentive person even without the Analysis.

But that case may by the by be taken notice of wherein the given perpendicular would not be BE but BF. For in this case the perpendicular BF being erected upon AB, the parallel FG would not cut the semi-circle; which is an infallible sign that the Problem in this case is impossible, where the perpendicular is supposed to be greater than half the base; which is inconsistent with a right angle.

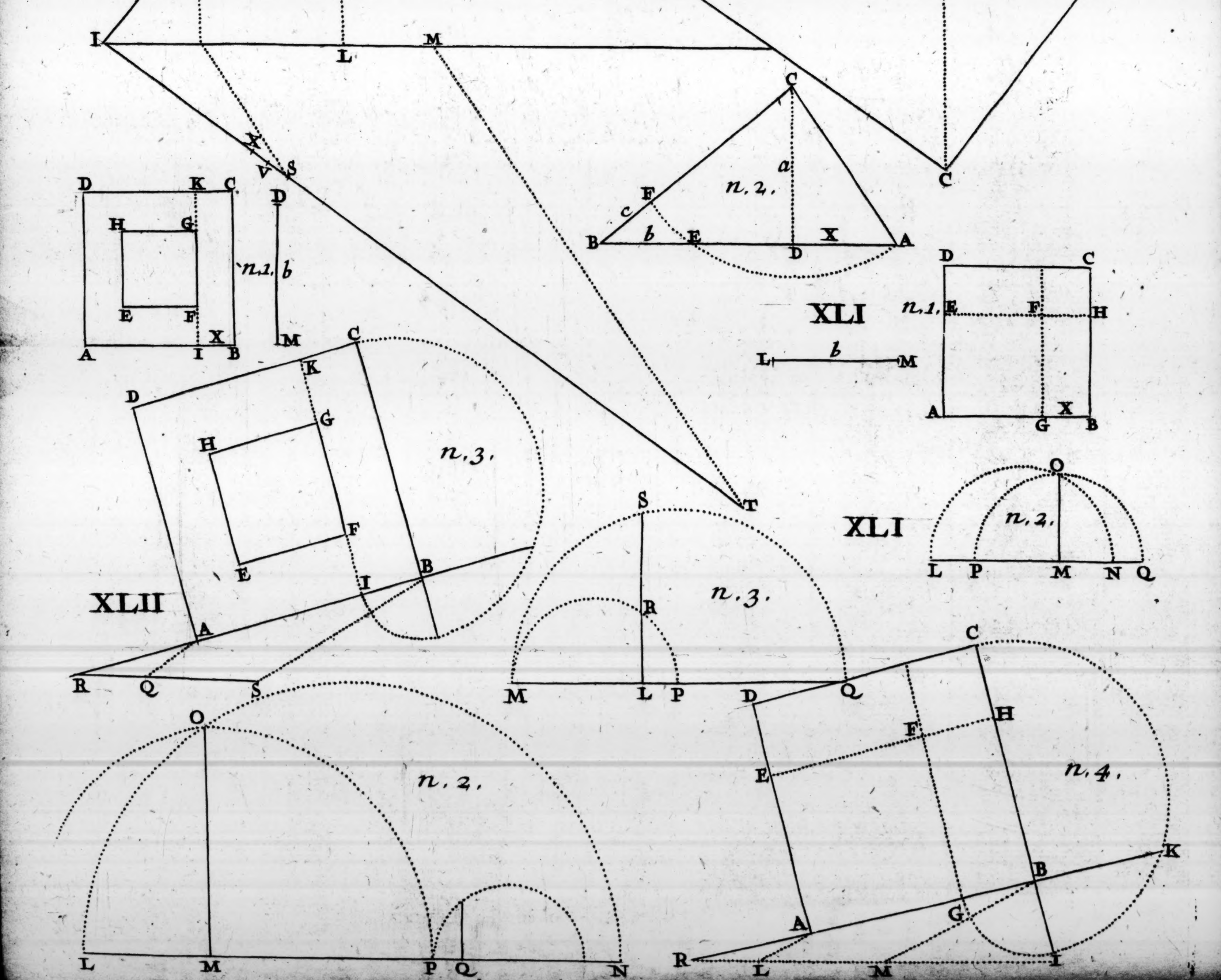
The Arithmetical Rule. From the square of half the base take the square of the given perpendicular, and add or substract the square root extracted out of the remainder, to or from half the base; and on the one hand the sum will give the greater segment, and on the other the difference will give the less.

PROBLEM VIII.

Having given the perpendicular of a right-angled Triangle that is to be let fall from the right angle, and the difference of the segments of the base, to find the segments, and describe the Triangle.

E. g. If the perpendicular is, as above, BE, and the difference of the segments AH (*n.* 1. *Fig* 38.) to find the segments AD and DB, from whose common term you are to erect a perpendicular DG or DC to form the Triangle.

SOLU-



XLI
n.1.
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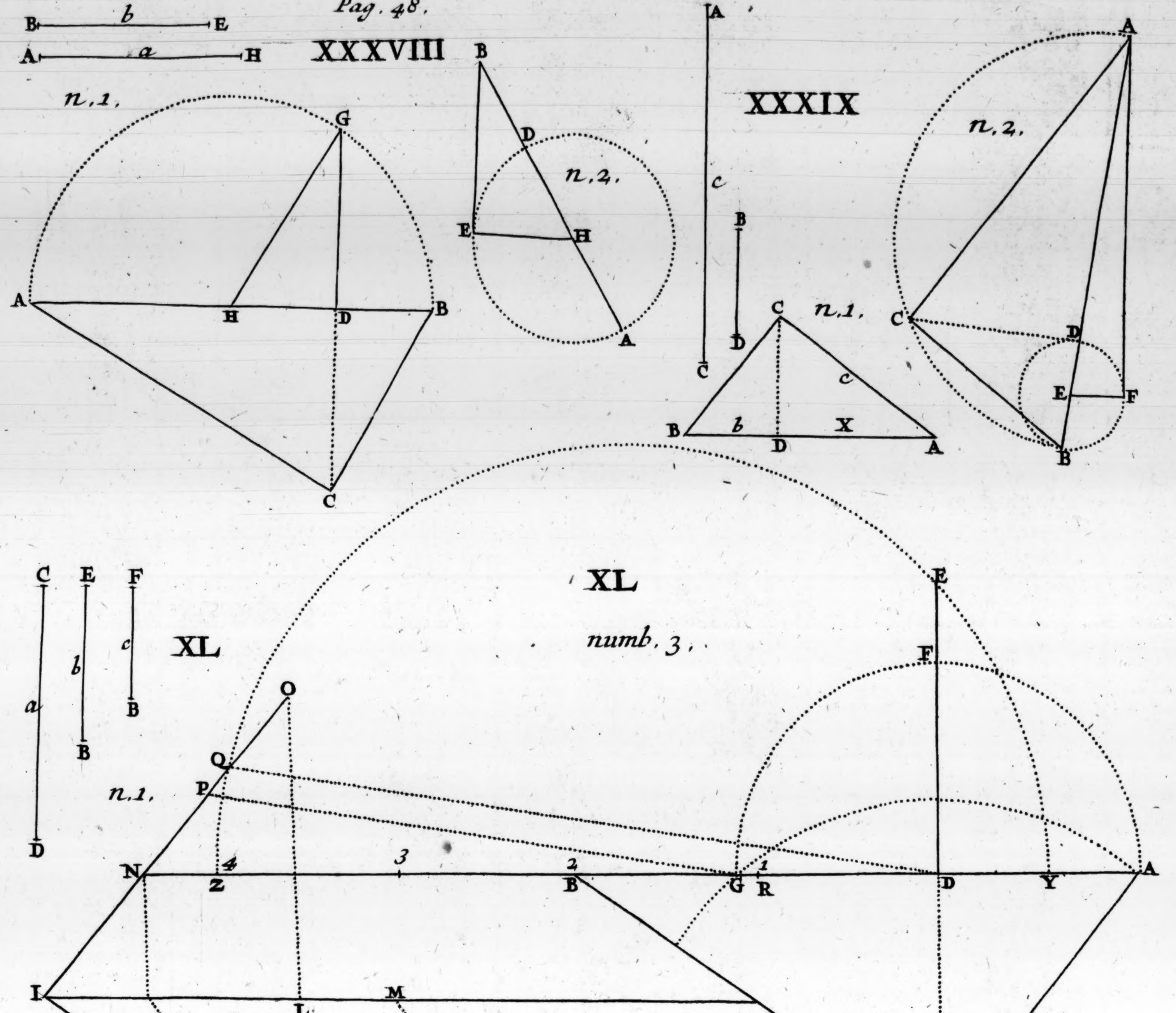
XXXVIII
n. 1.
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XXXIX
n. 1.
n. 2.
XL
n. 1.
XL
numb. 3.

SOLUTION.

Make the lesser segment $= x$, and the difference of the segnts $= a$, the greater segment will be $x + a$ Make the gi- perpendicular as before $= b$: Therefore you'l have

as $x + a$ to b so b to x; and consequently,

$xx + ax = bb$; and substracting ax,

$xx = bb - ax$. Wherefore according to case 2.

$x = -\frac{1}{2}a + \sqrt{\frac{1}{4}aa + bb}$.

The Geometrical Construction. Make HD $= \frac{1}{2}a$, DG equal d perpendicular to b; HG will be $= \sqrt{\frac{1}{4}aa + bb} =$ HB or A, *viz* having drawn a semi-circle from H thro' G. herefore DB is the less segment, and AD the greater; and ving drawn AG and BG, or on the other side, (making the pendicular DC $=$ DG) having drawn AC and BC, the iangle will be constructed. Or with *Cartes*, make (*n.* 2.) E $= \frac{1}{2}a$ and EB $= b$, and having described a Circle from thro' E draw BHA; and so you'l have the two segments ght AB the greater and DB the lesser.

The Arithmetical Rule. Join the squares of the half diffece and perpendicular into one sum, and then having extrad the root substract half the difference from it; and the reainder will be the lesser segment sought; and having added difference you'l have also the greater.

PROBLEM IX.

Having given for a right-angled Triangle one segment of the base and the side adjacent to the other segment, to find rest and construct the Triangle.

As if the lesser segment of the base DB be given (*Fig.* 39. 1.) and the side AC adjacent to the other segment; to find greater segment of the base, which being found the rest easily obtain'd, and consequently the whole Triangle.

SOLU-

SOLUTION.

Make the greater ſegment $= b$, the given ſide $= c$, the ſegment ſought $= x$. Now if we ſuppoſe the triangle ABC to be already found, it is evident, 1. If from the ſquare of AC you ſubſtract the $\square$ AD, you'l have the $\square$ CD $= cc - xx$. 2. The ſame $\square$ CD may alſo be otherwiſe hence obtain'd, becauſe, the angle at C being a right one, CD is a mean proportional between BD and DA, *i. e.* between b and x; whence the rectangle of the extremes bx is $= \square$ of the mean CD. Wherefore now it follows, 3. that

$cc - xx = bx$; and adding xx,
$cc = bx + xx$; and ſubſtracting bx,
$-bx + cc = xx$. Therefore according to caſe 2.
$x = -\frac{1}{2}b + \sqrt{\frac{1}{4}bb + cc}$.

Geometrical Conſtruction. Join EF $= \frac{1}{2}b$ (*n.* 2. *Fig.* 39.) and FA $= c$ at right angles, and having deſcribed a Circle from E thro' F draw AEB; ſo you'l have DA the greater ſegment and DB the leſs; having erected therefore a perpendicular from D, and deſcribed a ſemi-circle upon AB, you'l have C the vertex of the triangle ſought, whence you are to draw the ſides AC and BC.

The Arithmetical Rule. Join the $\square$ of half the given ſegment, and the $\square$ of the given ſide into one ſum; and having extracted the root of it, if you thence take half the given ſegment, you'l have the ſegment ſought.

PROBLEM X.

Having given in an oblique angled Triangle the perpendicular height, and the difference of the ſegments of the baſe, and the difference of the other ſides, to find the ſides and form the triangle.

As, if the altitude CD be given (*n.* 1. *Fig.* 40.) and alſo the difference of the ſegments of the baſe EB, and the difference of the ſides FB (as is evident from the triangle ABC (*n.* 2.) conceived to be ſo formed beforehand) and you are to determine the baſe it ſelf and both ſides, *&c.*

SOLU-

SOLUTION.

Make the given perpendicular CD $= a$ (see *n.* 2. *Fig* 40.) EB $= b$, FB $= c$: For the lesser segment of the base AD put x, and the greater will be $x + b$. It is now evident that you may obtain the □ CB by the addition of the □ □ DC and BD, $aa + xx + 2bx + bb$; and the □ AC by the addition of the □ □ AD and DC, *viz.* $aa + xx$: So that the side AC will be $= \sqrt{aa + xx}$, and the side BC $= \sqrt{aa + xx + 2bx + bb}$. But since also this same side BC may be obtain'd by adding the difference c to the side AC, so that it shall be $= c + \sqrt{aa + xx}$: you'l have this Equation,

$$c + \sqrt{aa + xx} = \sqrt{aa + xx + 2bx + bb};$$ and squaring both sides,

$$cc + aa + xx + \sqrt{4ccaa + 4ccxx} = aa + xx + 2bx + bb;$$

and substracting from both sides $cc + aa + xx$,

$$\sqrt{4ccaa + 4ccxx} = 2bx + bb - cc;$$

and again squaring it,

$$4ccaa + 4ccxx = 4bbxx + 4b^3x - 4bcx + b^4 - 2bbcc + c^4;$$

and substracting from both sides $4ccxx$ (because c is less than b) and transposing the rest,

$$\begin{array}{l} 4ccaa - 4b^3x - b^4 \\ \quad + 4bccx + 2bbcc = 4bbxx - 4ccxx; \\ \quad\quad - c^4 \end{array}$$

and dividing by $4bb - 4cc$,

$$\frac{\begin{array}{l} 4ccaa - 4b^3x - b^4 \\ \quad + 4bccx + 2bbcc \\ \quad\quad - c^4 \end{array}}{4bb - cc} = xx.$$

i. e. dividing the affected quantities by 4 both above and underneath,

$$\frac{-b^3x}{+bccx} \over bb - cc$$

$$\frac{-b^3x + bccx}{bb - cc} + \frac{4ccaa - b^4 + 2bbcc - c^4}{4bb - 4cc} = xx$$

and actually dividing the former part by $bb - cc$

$$-bx + \frac{4ccaa - b^4 + 2bbcc - c^4}{4bb - 4cc} = xx.$$

Therefore

Therefore according to the ſecond caſe,

$$-\tfrac{1}{2}b + \sqrt{\tfrac{1}{4}bb + \frac{4ccaa - b^4 + 2bbcc - b^4}{4bb - 4cc}} = x;$$

or reducing $\frac{1}{4}bb$ to the ſame denomination with the reſt

$$-\tfrac{1}{2}b + \sqrt{\frac{b^4 - bbcc + 4ccaa - b^4 + 2bbcc - c^4}{4bb - cc}} = x;$$

and leaving out thoſe quantities that deſtroy one another

$$-\tfrac{1}{2}b + \sqrt{\frac{4ccaa + bbcc - c^4}{4bb - cc}} = x.$$

The Arithmetical Rule. Multiply four times the ſquare of FB by the ſquare of the perpendicular CD, and add to it the product of the ſquare of EB into the □ FB, and from the ſum ſubſtract the biquadrate of FB; and the remainder will be the firſt thing found. Then ſubſtract four times the ſquare of FB from four times the ſquare of EB; and the remainder will be the ſecond thing found. Laſtly, divide the firſt thing found by the ſecond, and from the quotient take half after having extracted the root: Thus you'l have the leſſer ſegment of the baſe ſought, *&c. E. g.* In numbers you may put 2 for FB, 4 for EB, 12 for CD.

As for the *Geometrical Conſtruction*, the quantity of the laſt Equation contain'd under the radical ſign will help us to this proportion,

as $4bb - 4cc$ to $4aa + bb - cc$ ſo cc to a fourth; or dividing all by 4,

as $bb - cc$ to $aa + \frac{1}{4}bb - \frac{1}{4}cc$; ſo $\frac{1}{4}cc$ to a fourth, which is $\frac{1}{4}$ of the quantity under the radical ſign. Aſſuming therefore the quantity c for unity, make (*n.* 3.) IK $= c$ IN and KL $= b$; NO will be $= bb$, and ſubſtracting OP $= cc$ (*i. e.* to unity) there will remain NP $= bb - cc$. In like manner IS and KM $= a$, ST will $= aa$; to which if you add SX $=$ $\frac{1}{4}$NO, and take thence XV $= \frac{1}{4}$ unity; TV will be $= aa + \frac{1}{4}bb - \frac{1}{4}cc$. Wherefore if you make NR equal to this TV, and PQ $= \frac{1}{4}$ of unity or cc, ſince NP is $= bb - cc$; by the rule of proportion there will come out DR $\frac{1}{4}$ of that quantity, which is under the radical ſign. Therefore this being taken four times will give DZ for the whole quantity; to which if you

you join Dy = to unity, and, having described a semi-circle upon the whole line YZ, erect the perpendicular DE; this will be the root of the said quantity, and taking hence moreover EF = $\frac{1}{2}b$, you'l have DE or DA the less segment of the base sought. Therefore adding GB = b to DG, DB will be the greater segment, and, having let fall the perpendicular DC = a, BC and AC will be the sides sought. Q. E. F.

IV. *Some Examples of Affected Biquadratick Equations, but like Affected Quadratick ones.*

PROBLEM I.

To *find a square ABCD (such as in the mean while we'll suppose n.* 1. *to be in Fig.* 41.) *from which having taken away another square AEFG, which shall be half the former, there will be left the Rectangle GC whose Area is given.* E. g. Suppose the given area equal to the square of the given line LM, to find the true sides of the squares AB and AE, answering to these supposed ones, *n.* 1.

SOLUTION.

Make the area of the rectangle that is to remain = bb, and GB = x; BC or AB will be = $\frac{bb}{x}$, and substracting hence GB, the remaining side of the lesser square AG = $\frac{bb}{x} - x$, i. $\frac{bb - xx}{x}$. Since therefore the square of this is supposed to be half of the square of AB, this will be the Equation:

$$\frac{b^4 - 2bbxx + x^4}{xx} = \frac{b^4}{2xx};$$

and multiplying by xx,

$$b^4 - 2bbxx + x^4 = \frac{b^4}{2}$$

and multiplying by 2,

$$2b^4 - 4bbxx + 2x^4 = b^4;$$

and ſubſtracting $2b^4$, and adding $4bbxx$,

$$2x^4 = 4bbxx - b^4;$$

and dividing by 2,

$$x^4 = 2bbxx - \frac{b^4}{2}.$$

NB. The ſame Equation may be obtain'd, if, putting x for GB or FH, and having found the □ of AG or GF as above, you infer

$$\frac{b^4 - 2bbxx + x^4}{xx} = 2bb - xx.$$

This laſt Equation, tho' it be a biquadratick, yet may be rightly eſteem'd only a quadratick one, becauſe there is neither x^3 nor ſingle x in it, and ſo you may ſubſtitute this for it, $yy = 2bby - \frac{b^4}{2}$, *viz.* by ſuppoſing $y = xx$. Whence according to the third caſe of affected quadraticks,

$$y \text{ will} = bb + \sqrt{b^4 - \frac{b^4}{2}} \text{ i.e. } \sqrt{\frac{b^4}{2}}$$

$$\text{or} = bb - \sqrt{\frac{b^4}{2}}$$

Therefore $x = \sqrt{bb + \sqrt{\frac{b^4}{2}}}$.

Geometrical Conſtruction. Now if the given line b be aſſumed for unity, bb and b^4 will be $=$ $=$ to the ſame line. Therefore, if between LM as unity, and MN $= \frac{1}{2}b$ viz $\frac{b^4}{2}$ you find a mean proportional MO (*n.* 2. *Fig.* 41.) that will be $= \sqrt{\frac{b^4}{2}}$, which being ſubſtracted from LM, and added to it, will give the two values of the quantity y. Moreover therefore by extracting its roots, *i. e.* by finding other mean proportionals LR and LS between the quantities found LP and LQ and unity (*n.* 3.) they will be the two values of the quantity x ſought; the firſt whereof LR will ſatisfie the queſtion, and the other LS be impoſſible. Wherefore to form the

the square it self, since its side will be $= \frac{bb}{x}$; by making (n 4.) as x to b so b to a fourth, it will be obtain'd: And this may be further prov'd, if finding a mean proportional BK between BI = LR and the side of the □ BC, it be equal to the given quantity LM.

Arithmetical Rule. From the given area or the square of the given line LM substract the root of half the biquadrate of the same line; thus you will have the value of the □ FC, viz. xx: Therefore extracting further the square root of this, it will be the value of x sought.

PROBLEM II.

To find another square ABCD (Fig. 42. n. 1.) out of the middle whereof if you take another square EFGH, which shall be a fourth part of the former, the area of the rectangle BK intercepted between BC and FG prolonged, shall be equal to the square of a given line LM; i. e. having these given to find the segment BI, and consequently also the side BC or AB.

SOLUTION.

Make the area of the given rectangle, or the square of LM $= bb$, and the side sought of the rectangle BI $= x$; the other side BC will be $= \frac{bb}{x}$, and having substracted out of it AE and GK (*i. e.* $2x$) the side of the lesser square FG will be $\frac{bb}{x} - 2x$, i. e. $\frac{bb - 2xx}{x}$; whose square since it is the fourth part of the greater square by the Hypothesis, you'l have

$$\frac{b^4 - 4bbxx + 4x^4}{xx} = \frac{b^4}{4xx};$$

and multiplying both sides by xx,

$$4b^4 - 16bbxx + 16x^4 = b^4;$$

and taking away $4b^4$, and adding $16bbxx$,

$$16x^4 = 16bbxx - 3b^4;$$

 and

and dividing by 4,

$$x^4 = 4bbxx - \frac{3}{4}b^4.$$

Therefore according to the third case of affected quadratick E-quations,

$$xx = 2bb \pm \sqrt{4b^4 - \frac{3}{4}b^4} \text{ i. e.}$$

$$= 2bb \pm \sqrt{3\frac{1}{4}b^4}.$$

Therefore $x = \sqrt{2bb \pm \sqrt{3\frac{1}{4}b^4}}$.

Geometrical Construction. If the given line b be taken for unity, b^4 and bb will be equal to it. Therefore if between LM as unity, and MN $= 3\frac{1}{4}b$, you find a mean proportional MO (*n.* 2. *Fig.* 42.) 'twill be $\sqrt{3\frac{1}{4}b^4}$; which substracted from MQ $= 2b$, or added to it, will give two values of the quantity xx, *viz.* PQ and IQ; the first whereof will be only a true one, and of use here. Now therefore a mean proportional QR found between PQ and unity will express the quantity sought x.

Therefore for forming the square it self, since its side AB is $= \frac{bb}{x}$, you may proceed as in the former Construction, (*vide* *n.* 3.)

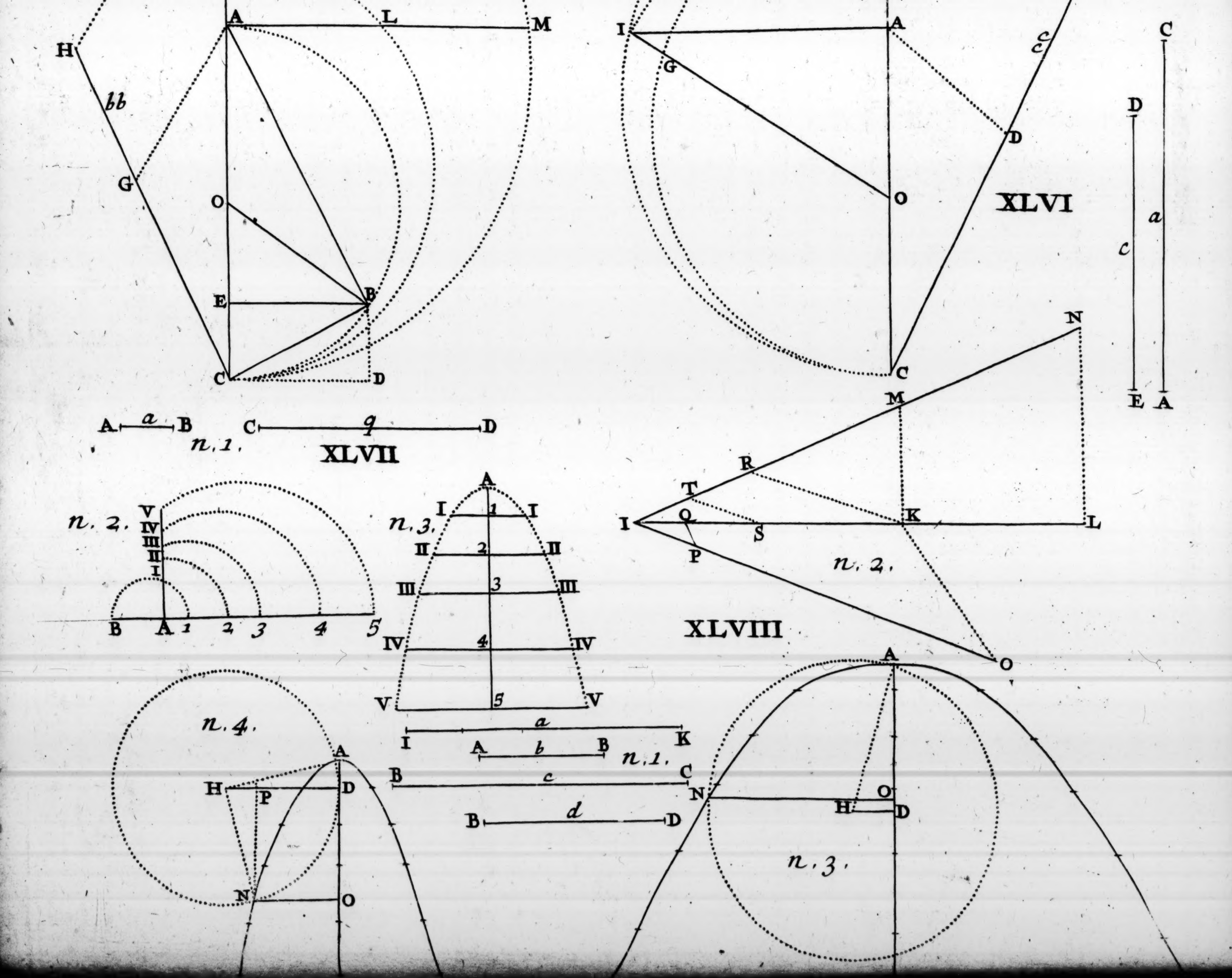
PROBLEM III.

H*Aving given the base of a right-angled Triangle, and a mean proportional between the Hypothenusa and Perpendicular, to find the Triangle.* As if the given base be AB (*Fig* 43.) and the mean proportional between AC and BC be CD; to find the perpendicular BC, and Hypothenusa AC.

SOLUTION.

Make the given base $= a$, and the mean proportional $= b$, the perpendicular BC $= x$, then will the Hypothenusa be $\frac{bb}{x}$, the Hypoth, $- \frac{bb}{x}$.

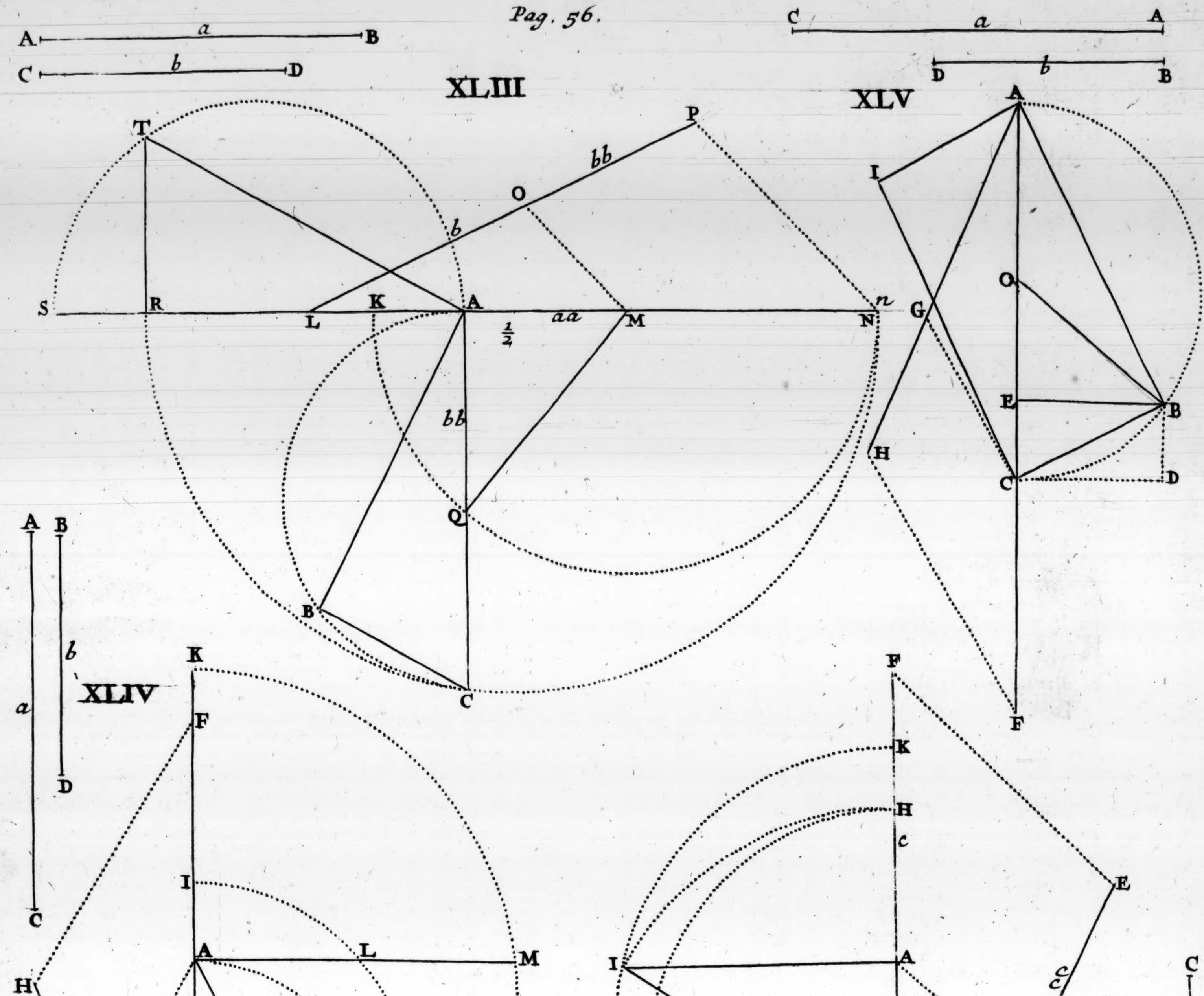
Theref



XLVI

XLVII

XLVIII


XLIII
XLIV
XLV

herefore
$\frac{b^4}{xx} - aa + xx$;
d multiplying both sides by xx,
$b^4 = aaxx + x^4$;
d substracting $aaxx$,
$b^4 - aaxx = x^4$.
herefore by the second case of affected quadraticks
$xx = -\frac{1}{2}aa + \sqrt{\frac{1}{4}a^4 + b^4}$,
and $x = \sqrt{-\frac{1}{2}aa + \sqrt{\frac{1}{4}a^4 + b^4}}$.

Or thus.

Make the Hypothenusa AC $= x$, then will the perpendi-
lar be BC $= \frac{bb}{x}$. Therefore
$xx = aa + \frac{b^4}{xx}$;
d multiplying by xx,
$x^4 = aaxx + b^4$.
herefore by case 1.
$xx = \frac{1}{2}aa + \sqrt{\frac{1}{4}a^4 + b^4}$
and $x = \sqrt{\frac{1}{2}aa + \sqrt{\frac{1}{4}a^4 + b^4}}$.

Geometrical Construction; the first for the latter Equation.
a be put for unity, the line AB will be also $= aa$, and
king, as a to b so b to a third, *i. e.* as LM to MN so LO
OP, and you'l have bb. Having erected the perpendicu-
AQ $=$ OP upon AM, and drawn MQ or Mn equal to
$\sqrt{\frac{1}{4}a^4 + b^4}$, and consequently A$n$ will be $= \frac{1}{2}aa + \sqrt{\frac{1}{4}a^4 + b^4}$,
e. the value of xx. Moreover a mean proportional AC
und between An and AR unity will be the value of x, *i. e.*
e Hypothenusa sought; which being found, you may easily
mplete the Triangle ABC.

2. In the case of the former Equation, making every thing
before AK would be the value of the quantity xx, *i. e.*
$-\frac{1}{2}a + \sqrt{\frac{1}{4}a^4 + b^4}$. Therefore a mean proportional RT
und between RS $=$ AK and AR unity will be the value of x,
i. e. the

i. e. the perpendicular ſought, and ſo AT the Hypothenuſa of the Triangle ſought.

Arithmetical Rule. In the firſt Solution add the biquadrate of the given mean proportional to the biquadrate of half the given baſe; and having extracted the ſquare root of the ſum, take from it half the ſquare of the given baſe; the root of the remainder will give the perpendicular of the triangle ſought, and the root of the ſum will give the hypothenuſa of it.

PROBLEM IV.

HAving *the Hypothenuſa of a right-angled Triangle given, and a mean proportional between the ſides to find the Triangle.* As if the hypothenuſa be AC (*Fig.* 44.) and a mean proportional between the ſides BD, to find the ſides AB and BC.

SOLUTION.

Make the given Hypothenuſa $= a$, and the mean proportional $= b$, and the perpendicular BC $= x$; the baſis AB by the hypoth. will be $\frac{bb}{x}$. Therefore

$$\frac{b^4}{xx} + xx = a;$$

and multiplying by xx,

$$b^4 + x^4 = aaxx;$$

and ſubſtracting b^4,

$$x^4 = aaxx - b^4.$$

Therefore by the third caſe,

$$xx = \tfrac{1}{2}aa \pm \sqrt{\tfrac{1}{4}a^4 - b^4}$$

$$\text{and } x = \sqrt{\tfrac{1}{2}aa \pm \sqrt{\tfrac{1}{4}a^4 - b^4}}.$$

Geometrical Conſtruction. If a be put for unity, AC will be alſo $= aa$, and by making as AC to CG (a to b) ſo AF to GH (b to a third) this third will be GH $= bb$. Aſſuming therefore OC $= \tfrac{1}{2}aa =$ OB the radius of a ſemi-circle, and having erected CD $= bb =$ BE parallel to it, EO will be

$\sqrt{\tfrac{1}{4}a^4} =$

$\sqrt{\frac{1}{4}a^4 - b^4}$, and consquently EC $= \frac{1}{2}a - \sqrt{\frac{1}{4}a^4 - b^4}$, and EA $\frac{1}{2}a + \sqrt{\frac{1}{4}a^4 - b^4}$, *viz.* the double value of the quantity xx. Therefore for the double value of x, you must extract the roots out of them, *i. e.* you must find the mean proportionals AL and AM between unity AC and AI $=$ EC on the one side, and AK $=$ AE on the other; Altho' these last may be more compendiously had, and the triangle it self immediately constructed, if having found EC and EA, you draw CB and AB: For these will be those two last mean proportionals $= =$ AL and AM; for by reason of the $\triangle\,\triangle$ ABC, AEB, and BEC, BC is a mean proportional between AC and CE, and AB a mean proportional between the same AC and AE by the 8. *Lib.* 6. *Eucl.* which is *Consect.* 3. *Schol.* 2. *Prop.* 34. *Lib.* 1. *Math. Enucl.*

PROBLEM V.

Having given the Area and Diagonal of a right-angled Parallelogram, to find the sides and so the Parallelogram. As if the given Area be $=$ to the square of a given line BD (*Fig.* 45.) and the Diagonal AC, to find the sides AB and BC.

SOLUTION.

If for the given Area, or square of the line BD you put bb, and make the Diagonal AC $= a$, and put for the lesser side BC, x; the other side will be $\frac{bb}{x}$.

Therefore $\frac{b^4}{xx} + xx = aa$;

and multiplying by xx,

$b^4 + x^4 = aaxx$; and substracting b^4,

$x^4 = aaxx - b^4$. Which Equation, since it is the same with that of the preceding Problem (which is no wonder, since this fifth perfectly coincides with the fourth; for the Diagonal AC is the hypothenusa, and BD, whose square is $=$ to the given area of the rectangle, is a mean proportional between the sides AB and BC) and so will have the same Construction,

struction, (see *Fig.* 45.) and the same Arithmetical Rule, which may be easily formed from the last Equation of the preceding Problem.

PROBLEM VI.

Having given the first of three proportional lines, and another whose square is equal to both the squares of the other two, to find those two proportionals. As if AC (*Fig.* 46.) be the first of the three proportionals, and another line ED given, whose square equals the two squares of the others taken together; to find those two as second and third proportionals.

SOLUTION.

If for AC you put a, and make the given line ED $= c$, and the second proportional $= x$, the third will be $\frac{xx}{a}$. Wherefore the squares of the two last will be $\frac{x^4}{aa} + xx = cc$, $\square$ ED by the hypoth. and multiplying both sides by aa.

$$x^4 + aaxx = aacc;$$

and substracting $aaxx$,

$$x^4 = -aaxx + aacc.$$ Therefore

$$xx = -\tfrac{1}{2}aa + \sqrt{\tfrac{1}{4}a^4 + aacc}$$

and $x = \sqrt{-\tfrac{1}{2}aa + \sqrt{\tfrac{1}{4}a^4 + aacc}}$.

Geometrical Construction. If a be put for unity, AC will also $= aa$ and a^4, and making as AC to CD (a to c) so AF to DE (c to a third) DE will be $= cc$. Now having made AK $=$ DE, *i. e.* cc, a mean proportional AI found between AC and AK will be $\sqrt{aacc}$. Therefore taking AO $= \tfrac{1}{2}$AC, *viz.* $\tfrac{1}{2}aa$, the hypothenusa OI will be $= \sqrt{\tfrac{1}{4}a^4 + aacc}$. And OA $= \tfrac{1}{2}a$ being substracted from OI or OH equal to it will leave AH the value of xx; and the root of that being extracted, *i. e.* finding another mean proportional AG between AC and AH, it will be the value of x, *i. e.* the second of the proportionals sought, and since AC is the first given, AH will be the third. Q. E. F.

NB. This

NB. This Construction may be abbreviated, and the first Operation, by which you find DE, to which afterwards AK is made equal, may be omitted. For since you make use of a mean proportional between CA and DE sought, which is = to the given line ED, and afterwards AI a mean proportional between AC and AK is sought; it is evident that AI will be equal to ED given, and consequently that they may be immediately joined at right angles at the beginning of the given line AC, and the rest may then be done as before.

Some Examples of Cubick and Biquadratick Equations, both simple and affected, whether reducible or not.

PROBLEM I.

Between two given right angles to find two mean proportionals. E. g. Suppose given AB the first and CD the fourth, (*Fig.* 47. *n.* 1.) between these to find two mean proportionals.

SOLUTION.

Make the first of the given quantities AB = a, the other CD = q, the first mean = x, then will the latter be $\frac{xx}{a}$, and consequently $\frac{x^3}{aa} = q$; and multiplying by aa, $x^3 = {*}{*}\, aaq$.

The Central Rule will b $\frac{L}{2} =$ AD

$\frac{r}{2L^2} =$ DH. i e. according to a supposition we shall by and by make, $\frac{1}{2}a =$ AD, and $\frac{1}{2}q =$ DH.

Geometrical Construction. If AB or a be made unity, and also the *Latus Rectum* of your Parabola, and you describe, by means of this *Latus Rectum*, the Parabola, according to *Schol.* 1. *Prop.* 1. *Lib.* 2. *Math. Enucl.* [see *n.* 2. and 3. *Fig* 47.] in which AB is the *Latus Rectum*; A1, A2, *&c.* the Abscissa's; AI, AH, *&c.* the semiordinates; make moreover (*n.* 4) AD = $\frac{1}{2}a$, and having from D erected a perpendicular = $\frac{1}{2}q$, describe a circle at the interval AH, cutting the para-

bola in N: Which being done, a perpendicular to the Ax will be the root ſought or the value of x, *i. e.* the firſt of the means, and conſequently OA the other; ſince NO by the firſt property of the Parabola (ſee *Prop.* 1. *Lib.* 2. *Matheſ. Enucl*) is a mean proportional between the *Latus Rectum* AB, and the abſciſſa AO. And by this means there will come out, by *Baker*'s Central Rule the very conſtruction of *Des Cartes, Geom.* p. m. 91.

The Arithmetical Rule. Multiply the ſquare of the firſt by the fourth given, and the cube root extracted out of the product, will expreſs the firſt of the means ſought.

PROBLEM II.

HAving given the ſolid Contents of a ſolid or an hollow Parallelepiped, and the proportion of the ſides, to find the ſides.* As, if the given capacity or ſolid contents be $=$ to the cube of a certain given line IK (*Fig.* 48. *n.* 1.) and the proportion of the heighth to the length be as AB to BC, and to the latitude as the ſame AB to BD; to find firſt the altitude, which being had, the other Dimenſions will alſo be known, by the given proportions.

SOLUTION.

Make IK $= a$, AB $= b$, BC $= c$, and BD $= d$; and laſtly the heighth ſought $= x$, then

as b to c, ſo x to the length required $\frac{cx}{b}$;

and as b to d, ſo x to the latitude ſought $\frac{dx}{b}$.

Multiplying therefore theſe three dimenſions of the Parallelepiped together, you'l have its capacity or ſolid contents $\frac{cdx^3}{bb}$

$= a^3$;

and multiplying by bb,

$cdx^3 = a^3bb$; and dividing by cd,

$x^3 = \frac{a^3bb}{cd}$ i e. $x^3 ** - \frac{a^3bb}{cd} = 0$.

There-

Therefore the Central Rule will be the same as above, $\frac{L}{2} = AD$ and $\frac{r}{2L^2} = DH$, *i. e.* according to the supposition which will by and by follow,

$\frac{a}{2} = AD$ and $\frac{a^3bb}{2cd} = DH.$

Geometrical Construction. If IK or a be made unity, and at the same time *Latus Rectum,* and by means of it you describe a Parabola, after the way we have shewn, *Fig.* 47. *n.* 2. and 3. and shall always hereafter make use of; and then to prepare the quantity $\frac{a^3bb}{2cd}$ (which in the Central Rule is the the quantity $\frac{r}{2}$) make (*n.* 2.)

IK—IM = BC—KL = BD—MN
as a to e so d to e;

so that for cd you may put ae, and afterwards divide by a both above and underneath; you'l have the quantity $\frac{r}{2} = \frac{aabb}{2e}$.

Therefore by further inferring
as $2e$ to bb, so aa to a fourth
IO—IP = IT—IK—IQ, and you'l have the quantity DH, which will determine the centre, after AD is made equal $\frac{a}{2}$. Having therefore described from that centre a circle through the vertex of the Parabola A (*n* 3.) a semiordinate NO drawn from the intersection will be the altitude sought, which will easily give you the length and breadth by the reasons above shewn.

Another Solution.

Which will be more accommodated to the Arithmetical Rule.

Let the rest of our Positions or *Data* remain as above, but the name of the proportion which the altitude has to the length be = e, and of that which it has to the latitude = i, the length

length will be $=$ to ex, and the latitude to ix. Wherefo[re] multiplying the sides together, you'l have the whole solidity

$$eix^3 = a^3; \text{ and dividing by } ei,$$

$$x^3 = \frac{a^3}{ei}. \text{ Therefore}$$

$$x = \sqrt{C.}\,\frac{a^3}{ei}. \text{ Hence}$$

The Arithmetical Rule. Multiply together the given nam[es] of the reasons, and divide the given cube by the produ[ct], which done, the cubick root extracted out of the quotient wi[ll] be the altitude of the solid sought.

Another *Geometrical Construction.* Now if we would al[so] construct this Equation $x^3 = \frac{a^3}{ei}$ geometrically, putting AB $=$ [a] for unity, BC and BD will be the names of the reasons $=$ [e] and i. Making therefore first

IK — IM — KL —— MN

as a to e so i to a fourth f (*Fig.* 49. *n* 1.) af wi[ll] be $= ei$, and the proposed Equation will have this form:

$$x^3 = \frac{a^3}{af} \text{ i. e. } \frac{aa}{f}.$$

Therefore 2. making

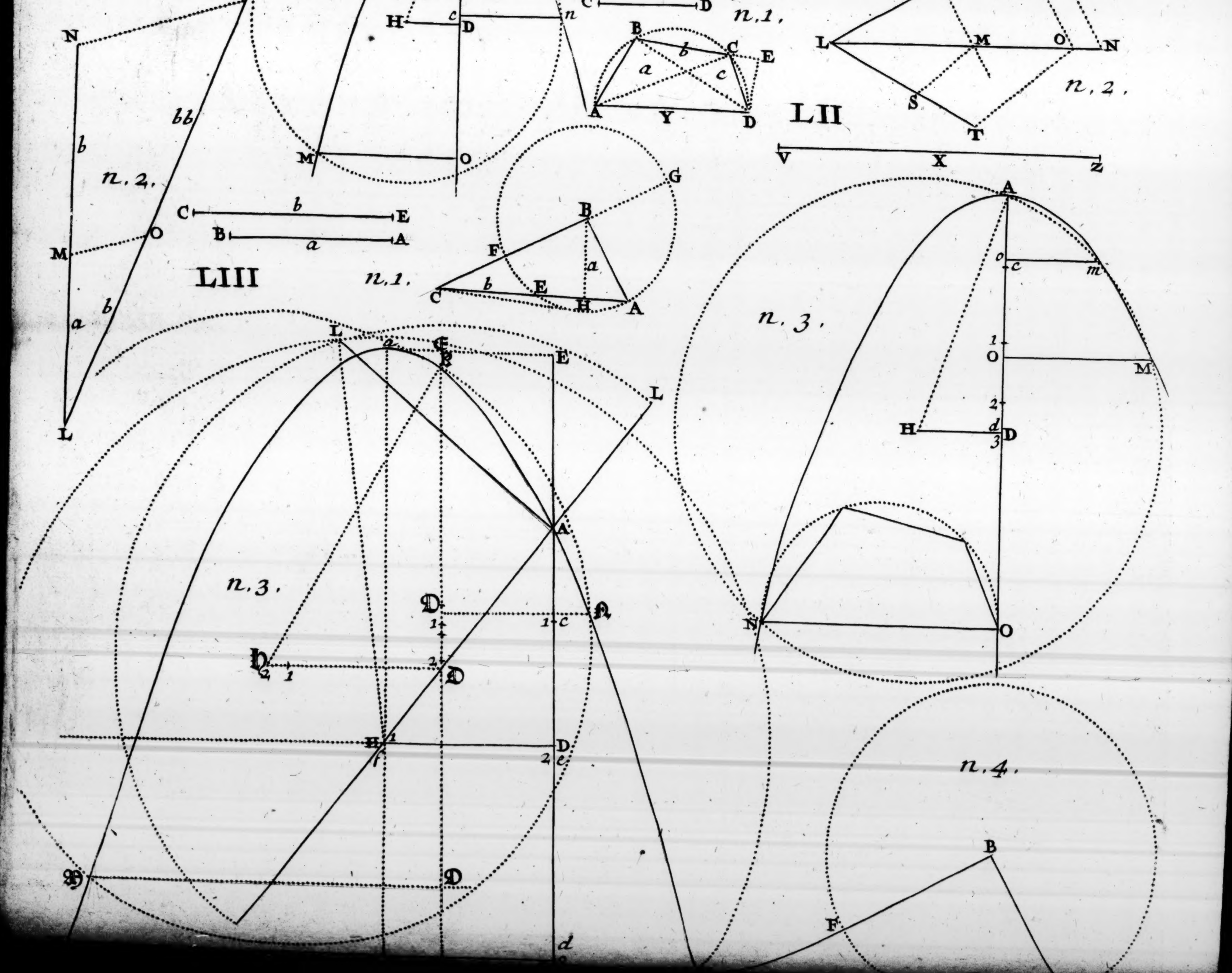
as f to a so a to a third IQ, that will be the value of x. But hence 3 by extracting the cube root, *i. e.* by finding two mean proportionals between unity b, *viz.* AB and the lin[e] found IQ; the first of them will give the root sought.

NB. The same Central Rule would come out according t[o] *Baker*'s Central Rule, which would have the same form of th[e] Equation, as the precedent Example $x^3 ** - aa = o$, i. [e.]

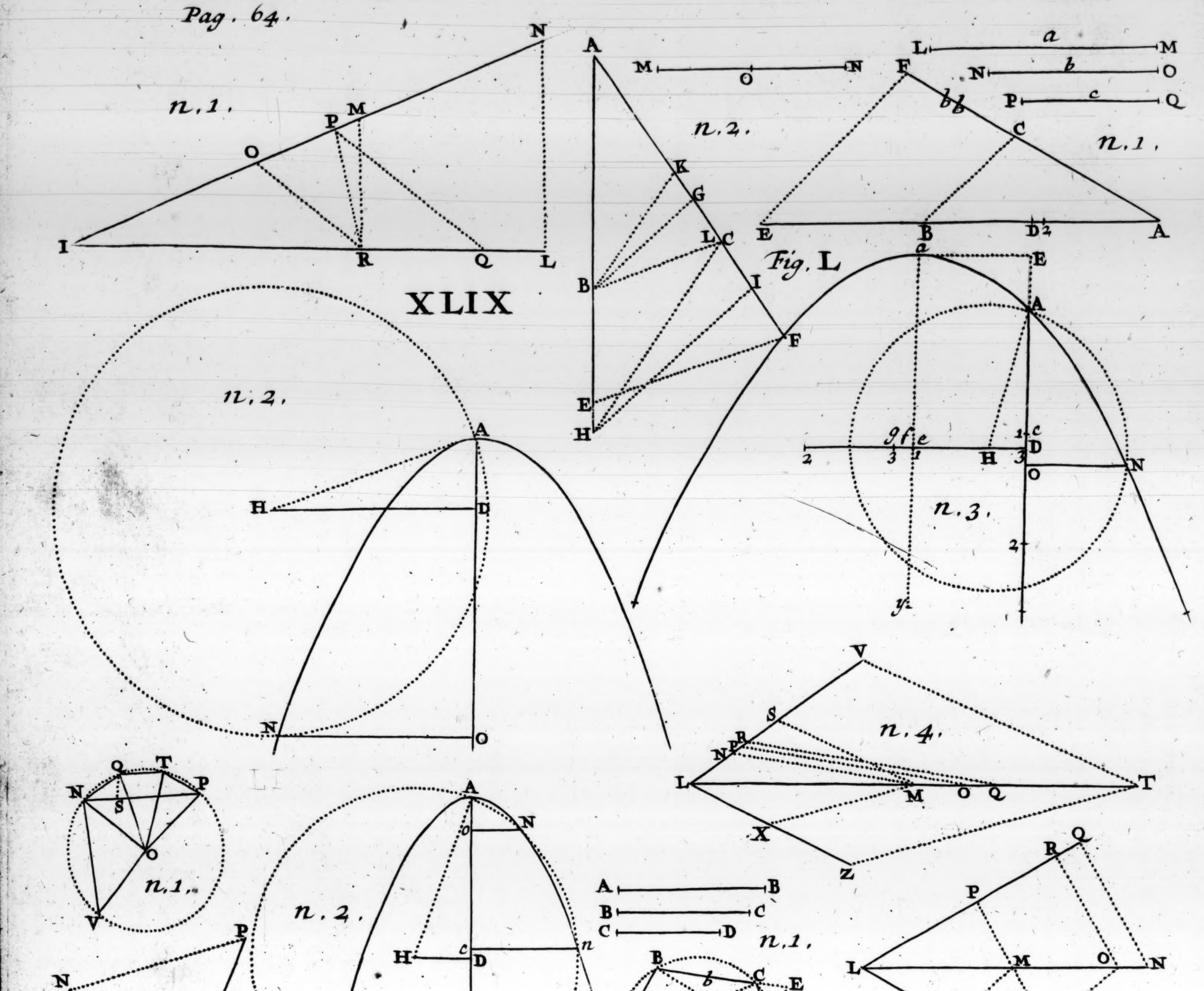
$$\text{taking } b \text{ for the } Lat. \atop Rect. \text{ and unity,}\ \left\{\begin{array}{l} \frac{L}{2} = \text{AD and } \frac{r}{2L^2} = \text{DH}, \\ \frac{b}{2} = \text{AD and } \frac{aa}{2f} \text{ i. e. } \tfrac{1}{2}\text{IQ} = \text{DH}. \end{array}\right.$$

(see *Fig.* 49. *n.* 2.)

PR



LII
n.1.
n.2.
n.3.
n.4.
LIII
n.1.
n.2.
n.3.



XLIX

Fig. L

PROBLEM III.

H Aving given the solid contents of a solid or hollow Paral-lelepiped, and the difference of the sides, to find the sides.

if the given capacity be equal to the cube of any given e LM (*Fig.* 50. *n.* 1.) and the difference whereby the gth exceeds the breadth $=$ NO, and the difference by which breadth exceeds the altitude or depth $=$ PQ, to find the gth, breadth and depth.

SOLVTION.

Make the side of the given cube $= a$, the excess of the gth above the breadth NO $= b$, and of the breadth above depth PQ $= c$, make the depth $= x$, the breadth $= x \dagger c$, d the length $= x \dagger b \dagger c$. Multiplying therefore these ee dimensions together.

$$
\begin{array}{ll}
\text{Length} & x \dagger b \dagger c \\
\text{Breadth} & x \dagger c \\
\hline
 & xx \dagger bx \dagger cx \\
 & \qquad\quad cx \dagger cb \dagger cc \\
\hline
 & xx \dagger 2cx \dagger bx \dagger cb \dagger cc \\
\text{Depth} & x \\
\hline
\end{array}
$$

$$
x^3 \left.\begin{matrix}\dagger 2c \\ \dagger b\end{matrix}\right\} x \left.\begin{matrix}\dagger cb \\ \dagger cc\end{matrix}\right\} x = a^3
$$

according to the forms of *Baker* and *Cartes*

$$
x^3 \begin{matrix}\dagger b \\ \dagger 2c\end{matrix}\, xx \begin{matrix}\dagger bc \\ \dagger cc\end{matrix}\, x - a^3 = o.
$$

Wherefore the Central Rule contracted by the supposition hich will hereafter follow will be this,

$$\frac{L}{2} \dagger \frac{p^2}{8} \backsim \frac{q}{2} = AD$$

nd $\frac{p}{4} \dagger \frac{p^3}{16} \backsim \frac{pq}{4} \backsim r = DH.$

e. by virtue of the supposition just now mentioned (which kes LM *viz.* *a* for unity and also for *Lat. Rectum*)

a †

$$\frac{a}{2} \dagger \frac{bb \dagger 4bc \dagger 4ce}{8} \backsim \frac{bc \backsim cc}{2} = \text{AD. And}$$

$$\frac{b \dagger 2c}{4} \dagger \frac{b^3 \dagger 6bbc \dagger 12bcc \dagger 8c}{16} \backsim \frac{bbc \backsim 3bcc \backsim 2c^3}{4}$$

$$(\backsim \frac{a^3}{2} = \text{DH.}$$

Or more short;

$$\frac{a}{2} \dagger \frac{bb}{8} \dagger \frac{bc \dagger cc}{2} \text{——} \frac{bc \text{——} cc}{2} \text{ i. e. } \frac{a}{2} \dagger \frac{bb}{8} = \text{AD; and}$$

$$\frac{b \dagger 2c}{4} \dagger \frac{b^3}{16} \dagger \frac{1\frac{1}{2}bbc \dagger 3bcc \dagger 2c^3}{4} \text{——} \frac{bbc \text{——} 3bcc \text{——} 2c^3}{4}$$

$$(\backsim \frac{a^3}{2} \text{ i. e.}$$

$$\frac{b \dagger 2c}{4} \dagger \frac{b^3}{16} \dagger \frac{bbc}{8} \backsim \frac{a^3}{2} = \text{DH.}$$

Geometrical Construction. If LM or a be taken for unity or *Latus Rectum* of the Parabola to be deſcribed, that being deſcribed (*Fig.* 50. *n.* 3.) you are firſt of all to determine two quantities AD and DH; which may be done two ways; either by *Baker*'s form of his Central Rule, or by ours immediately divided by the quantities of the laſt Equation.

1. For AD by our form, $\frac{a}{2} \dagger \frac{bb}{8} =$ AD, you muſt make (*Fig.* 50. *n.* 1.) as a to b ſo b to a third (AB to AC ſo BE to CF) which will be bb, and D^2 the eighth part of this CF muſt be added to A^2 the half of AB. And by *Baker*'s form you muſt make, 1. as AB ($= a$, *n.* 2.) to AC ($= \frac{1}{2}p$ i. e. $\frac{1}{2}b \dagger c$) ſo BE ($= \frac{1}{4}p$ i. e. $\frac{1}{4}b \dagger \frac{1}{2}c$) to a fourth CF (which will be $= \frac{p^2}{8}$:) 2. Make moreover as AB to AG (a to b) ſo BH to GI (c to bc) and, as AB to AK (a to c) ſo BH to KL (c to cc) and the two quantities found GI and KL (bc and cc) being added into one ſum will give the quantity $q =$ MN, the half whereof MO will expreſs the quantity $\frac{q}{2}$ in the

Rule,

Rule, and to be substracted from the former $\frac{a}{2} + \frac{p^2}{8}$. Actually therefore to determine the quantity AD not on the Ax, but on another Diameter of the described parabola *n.* 3. (because the quantity p is in the Equation) having made a perpendicular to the Ax aE $= \frac{p}{4}$ *i.e.* to the line BE *n.* 2. and from E having drawn EA parallel to the ax, according to our form AD *n.* 1. transfer it only on the diameter of the parabola *n.* 3. from A to D, either by parts $\frac{a}{2}$ i. e. $\frac{1}{2}$LM from A to c, and $\frac{bb}{8}$ i. e. $\frac{1}{8}$CF *n.* 1. from c to D: But according to *Baker's* form, first you must put $\frac{a}{2} = \frac{1}{2}$ LM *n.* 1. from A to 1. Secondly you must put from 1 to 2 the quantity $\frac{p^2}{8} =$ CF *n.* 2. Thirdly you must put from 2 to 3 backwards the quantity to be substracted $\frac{q}{2} =$ MO *n.* 2. which being done, the point D will be determined.

[*It is evident by comparing these two ways of Construction, that we may join our forms not incommodiously to* Baker's; *because by ours the quantity AD was obtained more compendiously than by* Baker's, *which will also often happen hereafter. And where this Compendium cannot be had, there is another not inconsiderable one, that, if the quantities AD and DH determined according to both ways shall coincide, (which happens in the present case) we may be so much the more sure of the truth*]

2. As for DH by our form, you must put it from D to e on a perpendicular erected from D on the left hand, the quantity $\frac{b+2c}{4} =$ BE *n.* 2. falling here upon the Ax. Then for the quantity $\frac{b^3}{16}$ make *n.* 4. as a to $\frac{1}{2}bb$ (LM to LN) so $\frac{1}{8}b$ to $\frac{b^3}{16}$ (MO to NP) and this quantity must be put from e to f in a perpendicular to the Diam. Thirdly, for the quantity $\frac{bbc}{8}$

you

you must farther make *n.* 4. as *a* to $\frac{1}{2}bb$ (DM to LN) so $\frac{1}{4}c$ to a fourth (MQ to NR) which must be put from *f* to *g*. Lastly the quantity $\frac{a^3}{2}$ (which is = MO *n.* 2.) must be put backwards (because to be substracted) from *g* to H, which is the centre sought. In like manner by *Baker's* form, first $\frac{p}{4}$ = BE is put from D to 1 even to the Ax. Secondly, for the quantity $\frac{p^3}{16}$ make, *n.* 4. as *a* to $\frac{p^2}{8}$ (LM to LS = CF *n.* 2.) so $\frac{p}{2}$ to a fourth (MT = AC *n.* 2. to SV) and this SV is further put (*n.* 3.) from 1 to 2. Thirdly, make in the same *Fig. n.* 4. for the quantity $\frac{pq}{4}$, as *a* to $\frac{p}{2}$ (LM to MT) so $\frac{q}{2}$ to a fourth (LX to XZ) and this XZ is put backwads (*n.* 3.) from 2 to 3, which precisely coincides with the point *g*. Lastly the remaining quantity $\frac{r}{2}$ (= MO *n.* 2. and so by what we have said above, precisely coinciding with the interval *g*H) is put backwards from *g* to H the Centre sought.

[*Hence it appears again that* Baker's *form is more laborious than ours; tho' both accurately agree, and hereafter, for the most part, we shall use them both together, tho in the work it self, rather in Figures, than in that tedious process of words, which we have here for once made use of, that it might be as an Example for the following Constructions.*]

Having therefore found by one or both ways the Center H, and thence described a circle thro' the vertex of the parabola A, the intersection N will give the perpendicular to the diameter NO, the value of the quantity *x* sought.

PRO-

PROBLEM IV.

TO *divide a given angle NOP, or a given arch NQTP (Fig. 51. n. 1.) into three equal parts*; *i. e.* having given or assumed at pleasure the Radius NO, and consequently also the Chord of the arch NP, to find NQ the subtense of the third part of the given arch.

SOLUTION.

If NO be made unity, NP $= q$, and NQ be supposed $= z$; having drawn QS parallel to TO, you'l have three similar triangles NOQ, QNR and RQS. For since the angle QOP is double of the angle QON, and the same (as being at the Center) double also of the angle at the Periphery QNR; it will be equal to the angle QON. But the angle at Q is common to both triangles: Therefore the whole are equi-angular, and consequently the legs NQ and NR equal, as also NO and QO; and by the like reason also PY and PT. Wherefore if RS should be added to RY, the line NP by this addition would be triple of the line NQ; and so would give the Equation, if RS was determined; which may be done by means of the △ QRS, similar to the two former NOQ and QNR; for the angle RQS is equal to the alternate one QOT $=$ QNR, and the angle at R common to the triangles QNR and RQS, *&c.* Wherefore

as NO to NQ so NQ to QR

$1 - z - z - zz$

and as NQ to QR so QR to RS

$z - zz - zz - \frac{z^4}{z}$

Therefore according to what we have above said

$q + z^3 = 3z$; and substracting q

$z^3 = 3z - q$; or

$z^3 - 3z + q = 0$

Therefore the Central Rule will be (supposing also unity NO for the *Latus Rectum*.)

$\frac{L}{2} + \frac{q}{2} = AD$ *i. e.* by our form, $\frac{1}{2} + \frac{3}{2}$ *i. e.* $2NO = AD$

and $\frac{r}{2} = DH$. and $\frac{q}{2} = DH$.

Geometrical Construction. Having described your parabola (*Fig.* 51. *n.*2.) take on its Ax (because the quantity *p* is wanting in the Equation) AD = 2NO, and from D having erected a perpendicular = $\frac{3}{2}$NP to H; that will be the Center, from which a circle described thro' A, by cutting the parabola in three places, will give the three roots of the Equation, *viz.* NO and *no* true ones, the first whereof will express the quantity sought NQ (*n.* 1.) the latter the line NV, being the subtense of the third part of the compl. of the arch; and MO will express the false root, which is equal to the former taken together: All the same as in *Cartes* p. 91. but here somewhat plainer and easier.

PROBLEM V.

Having three sides given of a quadilateral Figure to be inscribed in a circle, AB, BC, CD, (Fig. 52. n. 1.) to find the fourth side, which shall be the Diameter of the Circle.

SOLUTION.

If we consider the business as already done, and make AB = *a*, BC = *b*, CD = *c*, and AD = *y*; we shall have first in the right-angled △ ABD, □ BD = *yy* — *aa*, and (since in the obtuse angled △ BCD the □ BD is = □ □ BC + CD + 2 ▭ of BC into CE) if those two □ □ BC + CD (*i. e.* *bb* + *cc*) be substracted from □ BD (*yy* — *aa*) you'l have 2. *yy* — *aa* — *bb* — *cc* = to the two said rectangles of BC into CE. But these two rectangles may also be otherwise obtain'd, 3, if the segment CE be otherwise determined; which may be done by help of the similar △ △ ABD and CED (for the angles at B and E are right ones, and ECD and BAD equal, because each with the same third BCD makes two right ones;

ones; the one ECD by reason of its contiguity, the other at A by the 22. *Eucl. Lib.* 3.) *viz.* by inferring
as DA to AB so DC to CE
$y - a - c - \frac{ac}{y}$

for now multiplying CE $= \frac{ac}{y}$ by BC $= b$, the $\square$ of BC into CE will be $= \frac{abc}{y}$ and two such $\frac{2abc}{y}$.

Now therefore $yy - aa - bb - cc = \frac{2abc}{y}$; and multipl. by y,

$$y^3 * \left.\begin{matrix}-aa\\-bb\\-cc\end{matrix}\right\} y = 2abc;$$ i. e. by *Baker*'s and *Cartes*'s forms.

$$y^3 * \left.\begin{matrix}-aa\\-bb\\-cc\end{matrix}\right\} y - 2abc = o.$$

Therefore the Central Rule will be (supposing the same quantity *e. g.* a for unity and *Lat. Rect.*)
$\frac{L}{2} + \frac{q}{2} =$ AD and
$\frac{r}{2} =$ DH, *i. e.* according to our form,
$\frac{a}{2} + \frac{aa + bb + cc}{2}$ i. e. $a + \frac{bb + cc}{2} =$ AD, and

Geometrical Construction, which, without any circumlocution, from our form is founded on the foll. in *Fig.* 52. *n* 2.
LM $= a$ *n.* 3. A 1. $=$ LM from *n.* 2.
MN and LP $= b$ 1, 2 $= \frac{1}{2}$ PQ
PQ $= bb$ 2, 2 $= \frac{1}{2}$ ST
LS and MO $= c$ DH $=$ PR
PR $= bc$ MO and *mo* two false roots
ST $= cc$ NO the true root; upon which having described a semi-circle the quadrilateral will be easily made. According to *Baker*'s form, for AD there would be *n.* 3. first

$Ac = \frac{1}{2} LM$, then $cD = \frac{q}{2} = VX$, half the line VZ, which is compounded of LM, PQ and ST; but DH is = PR as above.

PROBLEM VI.

H*Aving given to form a right-angled Triangle the least side BA* (*Fig.* 53. *n.* 1.) *and the difference of the segments of the base, to find the difference of the sides, and so form the Triangle.* If we represent the business as already done, having given AB and EC to find FC.

SOLUTION.

Make $AB = a$, and $EC = b$, and $FC = x$; then will $BC = a + x$: Therefore the $\square\ AC = \overline{2aa + 2ax + xx}$ and the line $AC\ \sqrt{\overline{2aa + 2ax + xx}}$ and $\sqrt{\overline{2aa + 2ax + xx}} - b$. Now therefore ACE *i. e.* $\sqrt{\overline{2aa + 2ax + xx}}$ multiplyed by b or $\sqrt{bb}$ i. e. $\sqrt{\overline{2aabb + 2abbx + bbxx}}$ is = GCF [but GC is $= 2a + x$] i. e. $2ax + xx$ by *Cons.* 1. *Prop.* 47. *Lib.* 1. *Mathes. Enucl.* and squaring both sides

$$2aabb + 2abbx + bbxx = 4aaxx + 4ax^3 + x^4;$$

and transposing all,

$$x^4 + 4ax^3 + \begin{matrix} 4aaxx \\ -bb \end{matrix} - 2abbx - 2aabb = 0.$$

$$(p) \qquad (q) \qquad (r) \qquad (s)$$

Therefore (taking a for 1 and the *Latus Rectum* of the parabola) the Central Rule will be,

$$\frac{L}{2} + \frac{p^2}{8} - \frac{q}{2} = AD$$

and $\frac{p}{4} + \frac{p^3}{16} - \frac{pq}{4} - \frac{r}{2} = DH$ *i. e.* according to our form and Reduction,

$$\frac{a}{2} + \frac{16aa}{8} - \frac{4aa + bb}{2} \text{ i. e. } \frac{a}{2} + \frac{bb}{2} =$$

Geome-

Geometrical Conſtruction. Firſt from our form, the compendiouſneſs whereof will here appear, for it requires only one preparation *n.* 2. in which LM = a, MN and LO = b, OP = bb, which being premis'd, and the diameter Ay (becauſe the quantity p is in the Equation) being drawn (*n.* 3.) make AI = $\frac{1}{2}$LM, and 1, 2 or 1 D = $\frac{1}{2}$OP, and DH = LM. The reſt therefore being alſo perform'd, which the quantity S occurring in the preſent Equation requires, according to the laſt precepts of our introduction, you'l have NO the value of x ſought; whence (*n.* 4.) at the interval AB having deſcrib'd a Circle, and made a right angle at B, if FC be made = to the found NO, you'l have the △ ABC required, and EC will be found at the beginning of the preſcribed magnitude.

Now if you were to find the center H by *Baker*'s form without our Reduction, 1. you muſt put of (*n.* 3.) $\frac{1}{2}$AB from A to c. 2. For the quantity $\frac{p^2}{8}$ make as 1 to $\frac{p}{2}$ ſo $\frac{p}{4}$ to a fourth, which would be = 2AB, *viz.* 2DM, and to be ſet from c to d. 3. The quantity $\frac{q}{2}$ (to obtain which you muſt ſubſtract OP (*n.* 2.) from the quadruple of LM, and divide the remainder by 2) muſt be ſet off backwards from d to e, by thus determining the point D. 4. The quantity p, which here is preciſely = LM, muſt be transferr'd from D to f on a perpendicular erected from D. 5. For the quantity $\frac{p^3}{16}$ make as 1 to $\frac{p^2}{8}$ (= 2AB) ſo $\frac{p}{2}$ (alſo = 2AB) to a fourth quadruple of 2LM LM; and this muſt further be produc'd from f to the point g (which here the paper wil not permit) 6. For the quantity $\frac{pq}{4}$ make as 1 to $\frac{p}{2}$ ſo $\frac{q}{2}$ to a fourth, (which would be = to the quadruple of LM, but taking away OP) which muſt be ſet backwards from g to h. 7. Laſtly the quantity $\frac{r}{2}$ (= OP) ſet off from h backwards or towards the right hand to i will at length give the point H required.

Another

Another Solution of the ſame Problem.

This Problem may be more eaſily ſolved, and will give a far more ſimple Equation, if you are to find not FD but AE. Make therefore (*n.* 1. *Fig.* 53.) $= x$, and the reſt as above; AC will be $= x + b$, and its □ $xx + 2bx + bb$; therefore the □ BC $= xx + 2bx + bb - aa$; therefore the □ of the tangent HC $= xx + 2bbx + bb - 2aa$. But the rectangle ACE will be $bb + bx$. Therefore by *Prop.* 47. *Lib.* 1. *Math. Enucl.*

$$xx + 2bx + bb - 2aa = bb + bx;$$

and turning all over to the right hand,

$$xx + bx - 2aa = 0; \text{ or}$$

$$xx = -bx + 2aa.$$ Therefore by caſe 2 of affected quadraticks,

$$x = -\tfrac{1}{2}b + \sqrt{\tfrac{1}{4}bb + 2aa}.$$

The *Geometrical Conſtruction* may be performed according to the rules of quadraticks *Fig.* 54. *n.* 1. as will be evident to any attentive Reader. Having therefore deſcribed a circle at the interval BA, whether it be done from any arbitrary center (ſee *n.* 4 *Fig.* 53.) or upon AE found in the preſ. *Fig.* making an interſection at the ſaid interval in B; and applying AE, and producing it until EC become equal to the given quantity q, and at length having drawn BC you'l have the triangle right-angled at B, and alſo the difference of the ſides FC. But to make it more ſhort and elegant; having determined AE by a little circle (*Fig* 54 *n.* 3.) prolong it to the oppoſite part of the circumference in C, and draw AB and CB; for as the radius of the little circle is $\frac{1}{2}b$, ſo EC is $= b$.

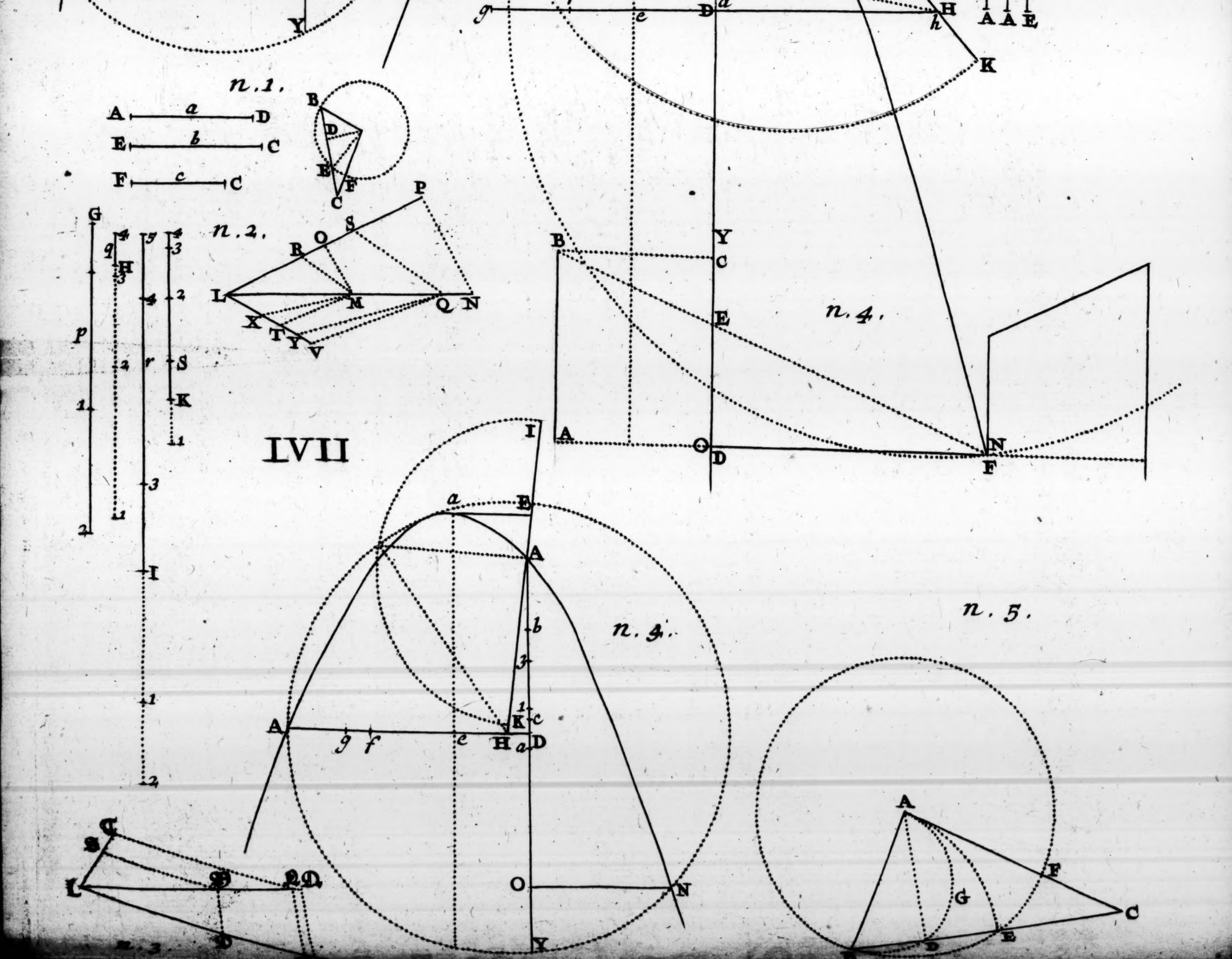
Now if you would conſtruct the Equation above by *Baker's* rule (that its univerſality may alſo be confirm'd by an Example in quadraticks) *viz.*

$$x^2 + bx - 2aa = 0;$$

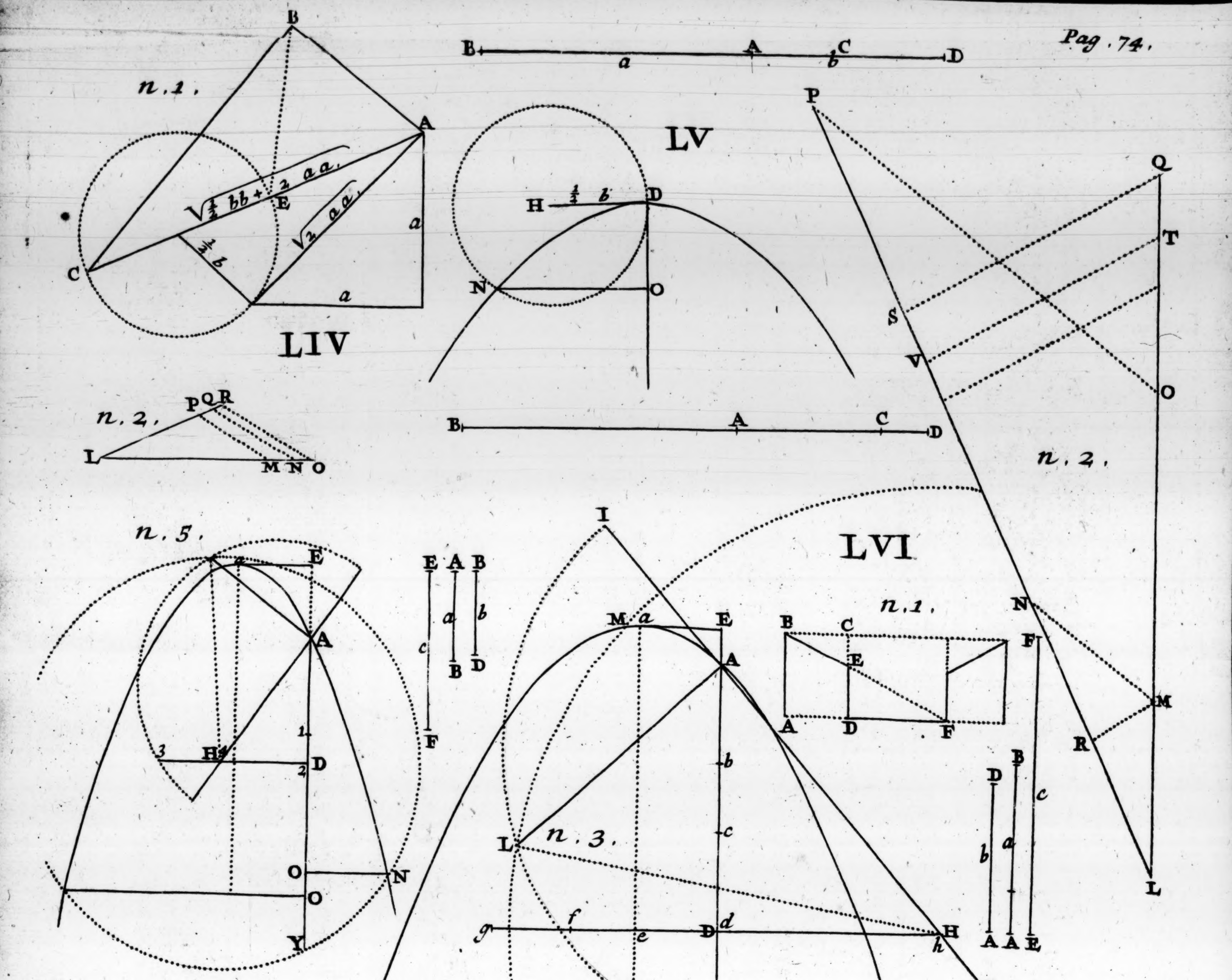
The central rule will be (taking a for 1 and the *L. R.*)

$$\frac{L}{2} + \frac{p^2}{8} + \frac{q}{2} = AD$$

and



LVII
n. 1.
n. 2.
n. 3
n. 3.
n. 4.
n. 5.


LIV
n. 1.
n. 2.
n. 5.
LV
LVI
n. 1.
n. 2.
n. 3.

nd $\frac{p}{4} + \frac{p^3}{16} + \frac{pq}{4} = DH$. *i. e.* by our Reduction,

$\frac{a}{2} + \frac{bb}{8} + aa$ i. e. $1\frac{1}{2}a + \frac{bb}{8} = AD$

nd $\frac{b}{4} + \frac{b^3}{16} + \frac{2aab}{4}$ i. e. $\frac{3b}{4} + \frac{b^3}{16} = DH$.

The *Construction* therefore will consist in these: LM (*n.* 2. *Fig.* 54) $= a$, MO $= \frac{1}{4}b$, LP $= \frac{1}{2}b$, therefore PR $= \frac{bb}{16}$: MN $=$ PR *viz.* $\frac{bb}{8}$, therefore Q $= \frac{b^3}{16}$. In *Fig.* 53. *n.* 3. C $= \frac{1}{4}b$, 𝔄 1 $= \frac{1}{2}$LM, 1, 2, or 1 𝔇 $=$ PR; 𝔇 1 $=$ LP MO, 1, 2, or 1 𝔓 $=$ PQ Having drawn a circle from thro' 𝔄 you'l have the true root ℜ𝔇 $=$ to AE sought; d 𝔐𝔇 the false root.

NB. Hence it is evident that one Problem may have several Solutions and Constructions, some more easy and simple, others more compound and laborious; *viz.* according as the unknown quantity is assumed more or less commodious to the purpose: which may not be amiss here to note for the sake of Learners.

PROBLEM VII.

SUppose a right line BD (Fig. 55.) any how divided in A, to divide it again in C, so that the square of BA shall be to the square of AC as AC to CD.

SOLUTION.

Since the first segments BA and AD are given, call the first a and the other b, and call AC the quantity AC x; and CD will be $= b - x$. Now therefore since we suppose

as the □ of AB to the □ AC so AC to CD

$$aa \text{ —— } xx \text{ —— } x \text{ —— } b - x$$

$aab - aax$ will $= x^3$, and transferring the quantities on the left hand to the right,

$$x^3 * aax - aab = o.$$

Where-

Wherefore the Central Rule (taking a for 1 and the *L. R.*) will be $\frac{1}{2}$—$\frac{q}{2}$ ⵘ AD

and $\frac{r}{2}$ ⵘ DH. *i. e.* by our form $\frac{a}{2}$—$\frac{aa}{2}$ i. e. o ⵘ AD, that D may fall on the vertex of the parabola;

and $\frac{aab}{2}$ i. e. $\frac{b}{2}$ ⵘ DH. The Construction therefore will be very simple, as is evident from the fifty fifth Figure.

PROBLEM VIII.

There is given AB the capital line of a horn work (which we represent (tho' rudely) n. 1. Fig. 56) and the Gorge AD, also part of the line of defence EF, to find the face BE, the flank DE, the curtain (or the chord) DF, also the angle of the Bastion ABE, &c. and so the whole delineation of the horn work. It is evident if you have the flank DE or the curtain DF, the rest will be had also. Suppose therefore, the capital line AB and the gorge AD, and part of the line of defence EF to be of the magnitudes denoted by the Letters a, b, c, on the right hand.

SOLUTION.

Make AB ⵘ a, AD ⵘ b, and EF ⵘ c, and DF ⵘ x; then will AF ⵘ $x + b$, and by reason of the similitude of the △ △ BAF and EDE and ECB,

as FA to AB so FD to DE

$$x+b \text{ —— } a \text{ —— } x \text{ —— } \frac{ax}{x+b}$$

But now □ □ DF + DE are ⵘ ⵘ □ EF i. e. $\frac{aaxx}{xx+2bx+bb}$ + xx ⵘ cc; or giving the same denomination to all the quantities on the left hand,

$$\frac{x^4 + 2bx^3 + aaxx \atop \qquad\qquad + bb}{xx + 2bx + bb} ⵘ cc,$$

and

and multiplying both sides by $xx \dagger 2bx \dagger bb$,

$$\begin{array}{l} x^4 \dagger 2bx^3 \dagger aaxx \sqsupset ccxx \dagger 2bccx \dagger bbcc\,; \\ \qquad\quad\;\; \dagger bb \end{array}$$

and translating all the quantities on the right hand by the contrary signs to the left,

$$\begin{array}{l} x^4 \dagger 2bx^3 \dagger aaxx - 2bccx - bbcc \sqsupset o. \\ \qquad\quad\;\; \dagger bb \\ \qquad\quad\;\; - cc \end{array}$$

Wherefore (putting a for 1 and *L. Rectum*) the Central Rule will be,

$\frac{L}{2} \dagger \frac{p^2}{8} \dagger \frac{q}{2}$ (because the quantity q is negative in the Equation, for cc is greater than $aa \dagger bb$) $\sqsupset$ AD;
and $\frac{p}{4} \dagger \frac{p^3}{16} \dagger \frac{pq}{4} - \frac{r}{2} \sqsupset$ DH. or according to our Reduction,

$\frac{a}{2} \dagger \frac{bb}{2} \dagger \frac{cc - aa - bb}{2}$ i. e. $\frac{cc}{2} \sqsupset$ AD

and $\frac{2b}{4} \dagger \frac{8b^3}{16} \dagger \frac{2bcc - 2aab - 2b^3}{4} - \frac{2bc}{2}$

i. e. $\frac{b}{2} \dagger \frac{b^3}{2} \dagger \frac{bcc - aab - b^3}{2} - bcc$ i. e.

$\frac{o - bcc}{2} \sqsupset$ DH.

Wherefore the *Geometrical Construction* requires no other preparatory determination by our form than of the quantities $\frac{cc}{2}$ for AD, $\frac{bcc}{2}$ for DH at the center, and $bbcc$ to determine the radius of the circle; which are exhibited by *n.* 2. *Fig.* 56. *viz* NP is $\sqsupset cc$, RS $\sqsupset bcc$, RV $\sqsupset bbcc$, which are found by means of LM $\sqsupset a$, LR $\sqsupset b$, LN and MO $\sqsupset c$, MQ $\sqsupset$ NP and MT $\sqsupset$ RS. Having therefore described a parabola, *n.* 3. and drawn its diameter, transfer AD $\sqsupset \frac{1}{2}$ NP upon it (because the quantity p is in the Equation) and also $\frac{1}{2}$RS from D upon H perpendicularly, and on the right hand, (because DH $\sqsupset - \frac{bcc}{2}$;) and so you'l have the center H;

thro' which having drawn KAI ſo that AK ſhall be $=$ to the quantity $bbcc$ or S, *i.e.* RV, *&c.* a circle deſcribed at the interval HL will cut the parabola in M and N, and applying the magnitude NO it will be that of the Curtain ſought; upon which, *n.* 4. having laid down the circumference of the horned work by help of the given lines AB and AD, you'l have the line EF, of the magnitude which was above ſuppoſed. Now if any one has a mind to do the ſame thing by *Baker*'s way; by laying down firſt the interval AB $= \frac{L}{2}$ and then making $bc = \frac{p^2}{8}$, and laſtly, putting cd for the quantity $\frac{q}{2}$; he will fall upon the ſame point D, and in like manner may expreſs the other parts of the Central Rule by the intervals De, ef, fg, and ſetting back the laſt gh, he will fall upon the ſame center H: But this is done with a great deal more trouble and labour to determine ſo many quantities, and alſo is in more danger of erring by cutting off ſo many parts ſeparately, as experience will ſhew; and thus we have by a new argument ſhewn the advantage of our Reduction.

Another Solution of the ſame Problem.

Things remaining as before (only aſſuming the given lines AB, AD and EF, *n.* 1. *Fig.* 56. one half leſs, that the Scheme may take up leſs room) make BE $= x$, as the firſt or chief unknown quantity; then will BF $= x + c$, and its $\square$ $xx + 2cx + cc$: And ſince

as BE to BC $=$ AD ſo BF to AF a fourth, which will be

$$x \text{ — } b \text{ — } x+c$$

$= \frac{bx + bc}{x}$ and its ſquare $= bbxx + \frac{2bbcx + bbcc}{xx}$. Wherefore if this ſquare be ſubſtracted from the ſquare of BF, there will remain the ſquare of BA, *i. e.*

$$xx + 2cx + cc - \frac{bbxx - 2bbcx - bbcc}{xx} = aa; \text{ i. e. all}$$

being reduced to the ſame denomination,

$$x^4 + 2cx^3 + \begin{matrix} ccxx \\ -bb \end{matrix} - 2bbcx - bbcc = aaxx;$$

or according to the forms of *Cartes* and *Baker*,

$$\begin{array}{l} x^4 + 2cx^3 + ccxx - 2bbcx - bbcc = o. \\ \qquad\qquad\quad -aa \\ \qquad\qquad\quad -bb \end{array}$$

Therefore the Central Rule (putting again a for 1 and the *L. R.*) will be

$$\frac{L}{2} + \frac{p^2}{8} - \frac{q}{2} = AD$$

and $\frac{p}{4} + \frac{p^3}{16} - \frac{pq}{4} - \frac{r}{2} = DH$;

or by our Reduction,

$$\frac{a}{2} + \frac{cc}{2} - \frac{cc + aa + bb}{2} \text{ i. e. } a + \frac{bb}{2} = AD;$$

and $\frac{c}{2} + \frac{c^3}{2} - \frac{c^3 + aac + bbc - bbc}{2}$ i. e.

$$c - \frac{bbc}{2} = DH.$$

Geometrical Construction. Having therefore described a parabola(*Fig* 56.*n*.5.)and drawn the diameter Ay,make A1 $= a$ and 1, 2 $= \frac{bb}{2}$, so you'l have the point D; make moreover D3 or 2, 3 $= c$, and backwards 3, 4 $= \frac{bbc}{2}$ (we here omit to express the Geometrical determination of these quantities $\frac{bb}{2}$ and $\frac{bbc}{2}$ as being very easy) and you'l have the point H, &c. and there will come out the quantity sought NO; which since it is equal to half BE *n.* 4. the business will be done; which *Baker*'s form will also give, exactly the same, but after a more tedious process.

PROBLEM IX.

IN *any Triangle ABC (the scheme whereof see n.* 1. *Fig.*57.) *suppose given the Perpendicular AD, and the differences of the least side from the two others EC and FC to find all the three*

sides. i. e. Chiefly the least side AB which being found, the others will be so also.

SOLUTION.

Make AD $= a$, EC $= b$, and FC $= c$, AB $= x$; then will BC $= x + b$ and its square be $xx + 2bx + bb$, and AC $= x + c$ and its square be $xx + 2cx + cc$; and BD will be $\sqrt{xx - aa}$, and DC $\sqrt{xx + 2cx + cc - aa}$. But the □ BC may also be obtain'd otherwise, and the Equation also, if □ □ BD + DC + 2 ▭ of BD by DC be added into one sum according to *Prop.* 4. *Lib.* 2. *Eucl.* viz.

$$2xx + 2cx + cc - 2aa + \sqrt{4x^4 + 8cx^3 + \left.\begin{matrix}4cc\\-8aa\end{matrix}\right\} xx - 8aacx + 4a^4 - 4aacc} \text{ will be } = xx + 2bx + bb,$$

[For DC $= \sqrt{xx - aa + 2cx + cc}$, multiplyed by BD $\sqrt{xx - aa}$, gives the rectangle of the segments

$$\sqrt{x^4 + 2cx^3 + \begin{matrix}ccxx\\-2aa\end{matrix} - 2aacx + a^4 - aacc - 2aa}$$

and this doubled *i. e.* multiplyed by $\sqrt{4}$, gives the quantity which is contain'd under the radical sign in the Equation]

Therefore turning all over on the left hand which are before the sign $\sqrt{\ }$ to the right hand, prefixing to them the contrary signs, you'l have

$$\sqrt{4x^4 + 8cx^3 + \left.\begin{matrix}4cc\\-8aa\end{matrix}\right\} xx - 8aacx + 4a^4 - 4aac}$$

$= - xx \begin{matrix}+2b\\-2c\end{matrix} x + bb - cc + 2aa$, and taking away the *Vinculum* on the left hand, and squaring on the right

$$4x^4 + 8cx^3 + \begin{matrix}4cc\\-8aa\end{matrix} xx - 8aacx + 4a^4 - 4aacc$$

$$= x^4 \begin{matrix}+4cx^3\\-4b\end{matrix} \left.\begin{matrix}-4aa\\+2bb\\+6cc\\-8bc\end{matrix}\right\} xx \left.\begin{matrix}+4b^3\\+4c^3\\+8aab\\-8aac\\-4bbc\\-4bcc\end{matrix}\right\} x \begin{matrix}+4a^4\\+b^4\\+c^4\\+4aabb\\-4aacc\\-2bbcc\end{matrix}$$

and

and adding and substracting on both sides, as much as can be,

$$3x^4 + 4cx^3 - 4aaxx = -b4x^3 \left.\begin{array}{l} +2bb \\ +2cc \\ -8bc \end{array}\right\} xx \left.\begin{array}{l} +4b^3 \\ +4c^3 \\ +8aab \\ -4bbc \\ -4bcc \end{array}\right\} x \begin{array}{l} +b^4 \\ +c^4 \\ +4aabb \\ -2bbcc \end{array}$$

and transferring all to the left,

$$3x^4 \left.\begin{array}{l} +4b \\ +4c \end{array}\right\} x^3 \left.\begin{array}{l} -4aa \\ -2bb \\ -2cc \\ +8bc \end{array}\right\} xx \left.\begin{array}{l} -4b^3 \\ -4c^3 \\ -8aab \\ +4bbcc \\ +4bcc \end{array}\right\} x \begin{array}{l} -b^4 \\ -c^4 \\ -4aabb \\ +2bcc \end{array} = o$$

and dividing all by 3,

$$x^4 \left.\begin{array}{l} +\frac{4}{3}b \\ +\frac{4}{3}c \end{array}\right\} x^3 \left.\begin{array}{l} -\frac{4}{3}aa \\ -\frac{2}{3}bb \\ -\frac{2}{3}cc \\ +\frac{2}{3}bc \end{array}\right\} xx \left.\begin{array}{l} -\frac{4}{3}b^3 \\ -\frac{4}{3}c^3 \\ -2\frac{2}{3}aab \\ +\frac{4}{3}bbc \\ +\frac{4}{3}bcc \end{array}\right\} x \begin{array}{l} -\frac{1}{1}b^4 \\ -\frac{1}{3}c^4 \\ -\frac{4}{3}aabb \\ +\frac{2}{3}bbcc \end{array} = o.$$

Note, I sought this Equation also after two other ways; 1 By a comparison of the □ AC with the two squares AB+BC, after 2 □ □ CBD thence substracted, according to *Prop.* 13. *Lib* 2.*Eucl* which is the 46. *Lib*.1. *Math. Enucl.* and I form'd the same with the present. 2. By putting at the beginning y for $x + b$ and z for $x + c$, and going on after the former methods, 'till you have this Equation,

$$x^4 \left.\begin{array}{l} -2x^2 \\ +4aa \end{array}\right\} yy \left.\begin{array}{l} -2x^2 \\ -2y^2 \end{array}\right\} zz \begin{array}{l} +y^4 \\ +z^4 \end{array} = o.$$

in which, when afterwards I substituted the values answering the quantities yy and zz, &c. This same last Equation came out a little easier, but (NB) with all the contrary signs.

Now to form the Central Rule, and thence make the Geometrical Construction, we must determine first each of the quantities p, q, r and S, that we may know whether they are negative or positive; and you'l find (*n.* 2. *Fig.* 57.) $p =$ G2 positive, $q =$ H^4 negative, and $K^4 = S$ also negative; and that by help of the quantities LM $= b$ or 1, MN and

LO

LO $= a$, OP $= aa$, MQ and LR $= c$, RS $= cc$, and alſo LT $= cc$, TV and LX $= c^3$, XY $= c^4$. Wherefore the form of the laſt Equation will be like this,

$x^4 + px^3 - qxx - rx - S = o$, and ſo the Central Rule (taking here b for 1 and the *L. R.*)

$$\frac{L}{2} + \frac{p^2}{8} + \frac{q}{2} = AD$$

and $\frac{p}{4} + \frac{p^3}{16} + \frac{pq}{4} - \frac{r}{2} = DH$.

Wherefore, having now deſcribed a parabola (as may be ſeen *n.* 4.) having found the diameter Ay transfer upon it firſt Ab $= \frac{1}{2}$ LM (*n.* 2) and then $bc = \mathfrak{DP}$ (*n.* 3.) i. e. $\frac{p^2}{8}$; and thirdly cD $= \frac{q}{2}$ i. e. $\frac{1}{2}$ H4 (*n.* 2.) moreover from D to e put off $\mathfrak{MB}$ (*n.* 3. $= \frac{p}{4}$, and from e to f put off $\mathfrak{DR}$ $= \frac{p^3}{16}$, and from f to g put off $\mathfrak{CS}$ $= \frac{pq}{4}$; and from g backwards to h put off half the quantity r, or I5 (*n.* 2.) and having done the reſt as uſual, you'l have NO, the ſide required of the Triangle to be deſcribed; the deſcription whereof will be now eaſy (*n.* 5.) having all the three ſides known. This may ſerve for an Examen, if having deſcribed a ſemi-circle AGB upon AB $=$ NO, you apply the given line AD, and from B thro' D draw indefinitely BDC: Then at the interval AB having deſcribed the Arches AE and BF, add the given line EC to BE, for thus having joined the points A and C, FC ought to be equal to FC before given.

SCHOLIUM.

WE have here omitted our Reduction, becauſe it would be too tedious, and would expreſs the quantities AD and DH (eſpecially the latter) in terms too prolix. For AD would be $= \frac{b}{2} + \frac{5}{9}bb + \frac{5}{9}cc - \frac{8}{9}bc + \frac{2}{3}da$ (*viz.* becauſe $\frac{p^2}{8}$ is found

found $\eqcolon \frac{2}{9}bb \dagger \frac{4}{9}bc \dagger \frac{2}{9}cc$ and $\frac{q}{2}$ taken in it self $\eqcolon \frac{2}{3}aa - \frac{1}{3}bb - \frac{1}{3}cc \dagger 43\,cc$; but here [where by vertue of the Central Rule it is taken positively, when it is in it self negative] under contrary signs it is $\eqcolon \frac{2}{3}aa \dagger \frac{1}{3}$ or $\frac{3}{9}bb \dagger \frac{1}{3}$ or $\frac{3}{9}cc - \frac{4}{5}$ or $\frac{12}{9}$ bc) or yet more contractedly (because b is unity) AD $\eqcolon \frac{1}{2}b \dagger \frac{5}{9}b$ (i. e. $1\,\frac{1}{18}b$) $\dagger \frac{5}{9}cc - \frac{8}{9}c \dagger \frac{2}{3}aa$; which parts may be expressed without any great difficulty on the Diameter Ay, by its portions A1, 1, 2; 2, 3; 3D: But the other quantity DH, or the definition of the Center H, would also have some tediousness, as because

$\frac{p}{4}$ would be $\eqcolon \frac{b \dagger c}{3}$

$$\frac{p^3}{16} \eqcolon \frac{4b^3 \dagger 12bbc \dagger 12bcc \dagger 4c^3}{27}$$

$$\frac{pq}{4} \eqcolon \frac{6bcc \dagger 6bbc - 4aab - 4aac - 2b^3 - 2c^3}{3}$$

or $\frac{6bbc \dagger 6bcc - 12aab - 6b^3 - 6b^3}{9}$

If you take away out of the quantities $\frac{pq}{4}$ and $\frac{r}{2}$ (since this latter is to be substracted, and so left, as it is, under the sign —; but the other, also negative in it self, but here positively expressed in the Central Rule, must have all the contrary signs) I say, if you take out of these quantities those which destroy one another, and add the rest with the two former quantities, they will be $\frac{b \dagger c}{3}$

$$- \frac{8aab \dagger 4aac \dagger 4bbc \dagger 4bcc}{9} - \frac{8b^3 - 8c^3}{27} \eqcolon \eqcolon \text{DH}.$$

or a little more contracted (because b is 1) $\frac{b \dagger c}{3}$

$$- \frac{8aa \dagger 4aac \dagger 4c \dagger 4cc}{9} - \frac{8b^3 - 8c^3}{27} \eqcolon \text{DH}.$$

But now if any one has a mind to illustrate this by a numeral Example and try the truth, *&c* of the quantities found; they may make *e. g.* $a \eqcolon 12$, $b \eqcolon 1$, and $c \eqcolon 2$; and they will

will easily find that in the last Equation the quantity p will be 4, and $q - 190$, $r - 388$, $S - 195$: Secondly in the Central Rule of *Baker* they'l find $\frac{L}{2} = \frac{1}{2}$, $\frac{p^2}{8} = 2$, and $\frac{q}{2} = 95$, and so the whole line AD $= 97\frac{1}{2}$; and further $\frac{p}{4} = 1$, $\frac{p^3}{16} = 4$, $\frac{pq}{4} = 190$, and $\frac{-r}{2} = -194$, and so the whole line DH $= 195 - 194$ *i. e.* $= 1$. Thirdly, likewise in our Reduction (if we proceed by each part corresponding to *Baker's*) $\frac{b}{2} = \frac{1}{2}$, $\frac{7}{9}bb + \frac{4}{9}bc + \frac{7}{9}bc = 2$, and $\frac{1}{3}aa + \frac{4}{3}bb + \frac{4}{3}cc - 4bc = 95$. The sum for AD $97\frac{1}{2}$; but further $\frac{b+c}{3} = 1$,

$$\frac{4b^3 + 12bbc + 12bcc + 4c^3}{27} = 4;$$

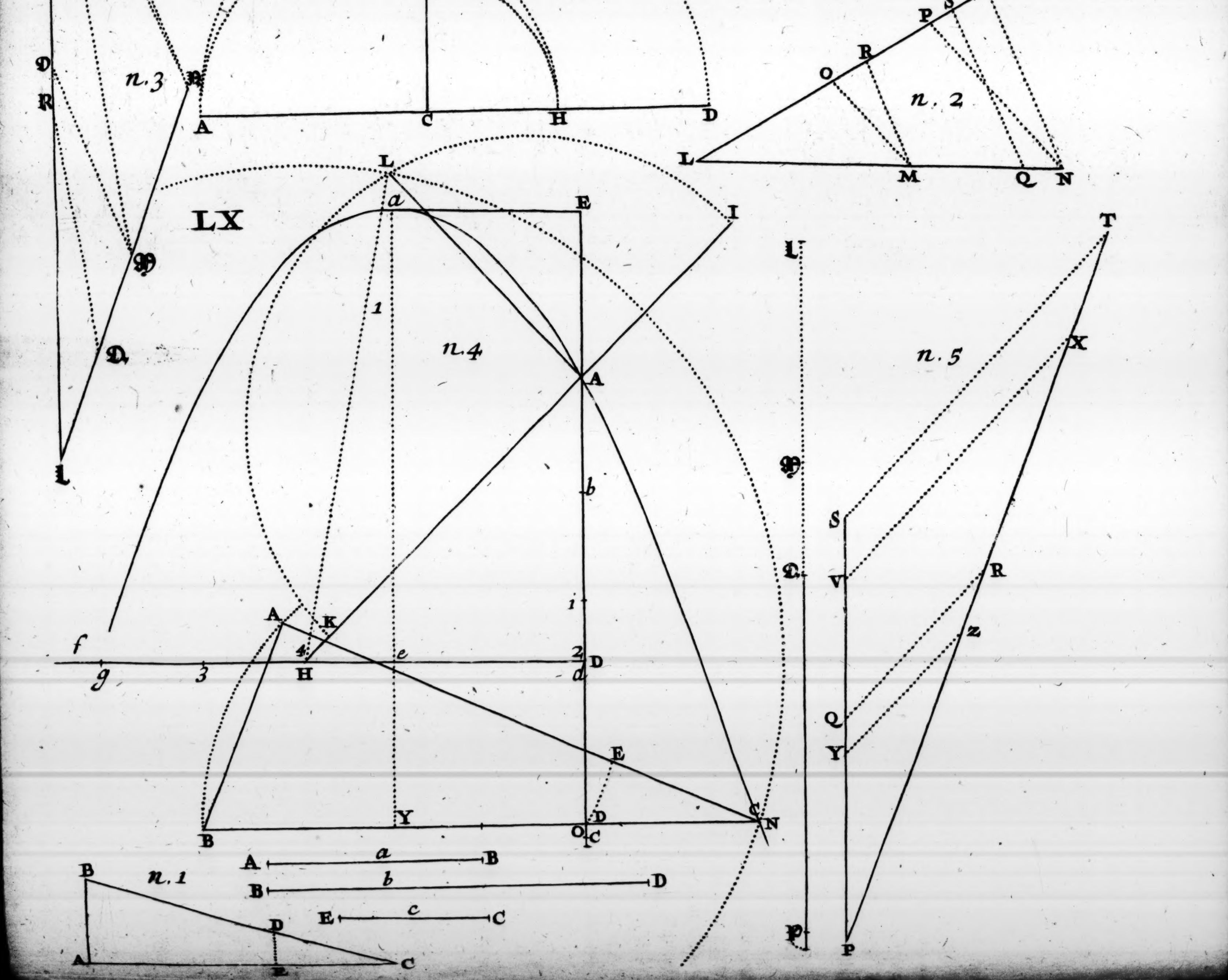
$$\frac{6bcc + 6bbc - 4aab - 2b^3 - 2c^3}{3} = 190;$$

and $2bbc + 2bcc - 4aab - 2b^3 - 2c^3 = 194$ to be substracted; and so the sum for DH $= 195 - 194 = 1$. Which same quantities will fourthly come out, if the quantities AD and DH contracted, as they are expressed in letters above, be resolved into numbers.

PROBLEM X.

YOU are to build a Fort on the given Polygons EAF (see Fig. 58. n. 1.) whose capital line AB shall equal the aggregate of the gorge and flank, and the squares of these added together shall be equal to the square of the given line GH, and the solid made by the multiplication of the square of the flank by the gorge, shall be equal to the cube of the given line IK.

SOLU-


LX
n.1
n.2
n.3
n.4
n.5

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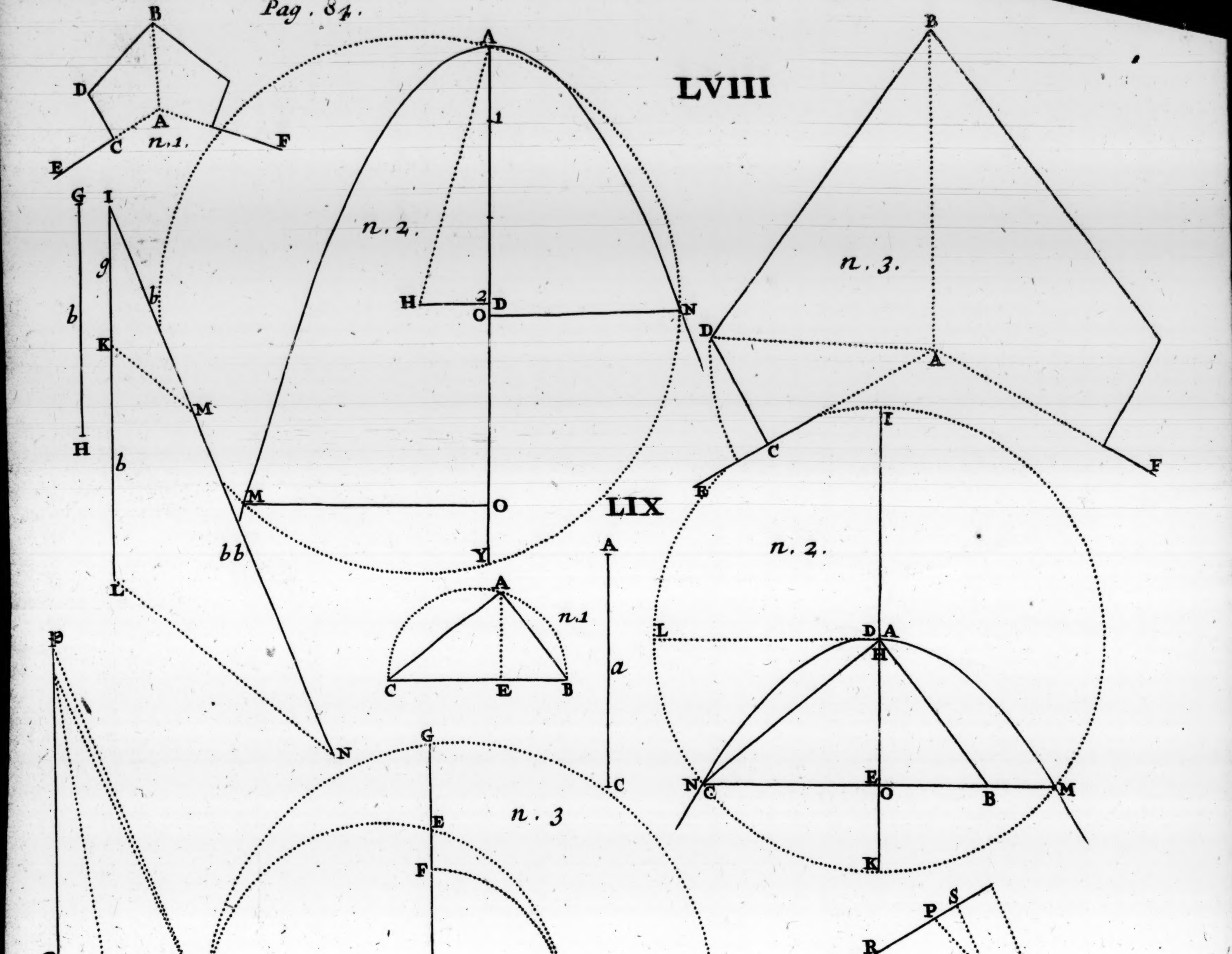

SOLUTION.

Make the Gorge AC $=$ z, whose square zz substracted from bb the square of the given line b, will leave the square of the flank DC $=$ $bb - zz$. Now this square being multiplyed by the gorge AC or z will give $bbz - z^3 = g^3$, the cube of the given line IK; and adding to both sides z^3, and substracting bbz, $o = z^3 * - bbz + g^3$.

Therefore if we take g or IK for 1 and L³, g^3 will be the line g, and

The Central Rule : $\frac{L}{2} + \frac{q}{2} = AD$.

$$\text{and } \frac{r}{2} = DH.$$

i. e. according to our Reduction,

$\frac{g}{2} + \frac{bb}{2} = AD$ and $\frac{g}{2}$ DH.

Geometrical Construction. Having described a Parabola (*n.* 2. *Fig.* 58.) make on its ax A1 $= \frac{1}{2}$IK and 1, 2, f. *viz.* 1D $= \frac{1}{2}$MN (from *n.* 1.) and DH $= \frac{1}{2}$IK. Then having described a circle from H, and found the true root NO upon the given angle EAF (*n* 3.) make AC $=$ NO, and having erected the perpendicular CD divide it by AD $=$ GH (*n.* 1.) and make AB $=$ AC $+$ CD; and the Fort will be drawn.

PROBLEM XI.

IN a right-angled Triangle ABC (which we denote by n. 1. Fig. 59.) having given the greater side AC, and made the less side AB = to the segment CE, which shall cut off from the base BC a perpendicular let fall from the right angle A; to find these lines AB or CE, and consequently the whole triangle.

SOLUTION.

Make AC $= a$, CE or AB $= x$. Therefore, 1. you'l have $aa - xx = \square$ AE. And becauſe the $\triangle\triangle$ BEA and CAE are ſimilar, you'l have

as AC to CE ſo BA to AE

$$a — x — x — \frac{xx}{a}$$

And ſo, 2. $\square$ AE $= \frac{x^4}{aa}$. Therefore

$\frac{x^4}{aa} = aa - xx$; and multipl. by aa,

$x^4 = a^4 - aaxx$; or, according to the form of *Cartes* and *Baker*, tranſpoſing all to the left,

$x^4 * + aaxx * - a^4 = o$ i. e.

$x^4 * + qxx * - S = o$.

Therefore (taking a for 1 and *L. R.*) the Central Rule will be $\frac{L}{2} - \frac{q}{2}$ i. e. $o =$ AD

and $\frac{r}{2}$ i. e. $o =$ DH; that H may fall on the vertex A.

Geometrical Conſtruction. Since a is aſſumed for unity and L, the quantity S alſo and *Latus Rectum* i.e. AI and AK and conſequently the mean proportional AL and the radius HL will be $= =$ to the given ſide AC, and conſequently at that interval having deſcribed a circle, thro' the Parabola rightly delineated, you'l have NO the value of the quantity x, *i.e.* of the leſſer ſide AB Having drawn therefore NA, which is $=$ to AC by Conſtruction, if you draw to it the perpendicular AB cutting NO produc'd to B, you'l have the Triangle ſought ABC, and AB (which will be a ſign of a true Solution) will be found $=$ NO or CE.

Another Conſtruction. Since in the Equation above found there is neither x^3 nor x, it may be look'd upon as a quadratick, and conſtructed after the ſame way, as ſeveral other like it among the Examples *n.* 4. *viz.* becauſe $x^4 = - aaxx + a^4$; according to caſe 2 of affected quadraticks,

xx

$xx = -\frac{1}{2}aa + \sqrt{\frac{1}{4}a^4 + a^4}$ i. e.
$+ \sqrt{\frac{5}{4}a^4}$,

and $x = \sqrt{-\frac{1}{2}aa + \sqrt{\frac{5}{4}a^4}}$ i. e. $\sqrt{-\frac{1}{2}a + \sqrt{\frac{5}{4}a}}$
Wherefore (*n.* 3. *Fig.* 59) if AC be made $= a$ and CD $\frac{5}{4}a$, the mean proportional CG will be $= \sqrt{\frac{5}{4}a}$, and taking hence GF $= \frac{1}{2}a$, there will remain FC $= \sqrt{\frac{5}{4}a} - \frac{1}{2}a$: And now between this or CH equal to it, and unity AC, having found another mean proportional CE it will be the value of the quantity x ſought $=$ NO (*n.* 2.)

PROBLEM XII.

IN a right-angled Triangle ABC (*Fig.* 60. *n.* 1.) *there is given the Perpendicular BA, a ſegment of the Hypothenuſa BD, and a ſegment of the Baſe EC, from C to the perpendicular DE let fall from the end of the ſegment BD; to find AE DC, and conſequently the Baſe AC and the Hypothenuſa BC, and ſo the whole Triangle.*

SOLUTION.

Make AB $= a$, BD $= b$, and EC $= c$, and DC $= x$; which being given, the reſt cannot be wanting: Therefore $xx - cc = \square$ DE. But the ſame $\square$ DE may be had, if you infer

as BC to BA ſo DC to DE

$b+x \text{ — } a \text{ — } x \text{ — } \dfrac{ax}{b+x}$

And then ſquare the quantity DE, the ſquare will be

$$\frac{aaxx}{xx+2bx+bb} = xx - cc;$$

and multiplying both ſides by $xx + 2bx + bb$,

$$aaxx = x^4 + 2bx^3 + bbxx - 2bccx - bbcc;$$
$$ -cc$$

and ſubſtracting alſo $aaxx$,

$$x^4 + 2bx + bbxx - 2bccx - bbcc = 0.$$
$$ -aa$$
$$ -cc$$

 Which

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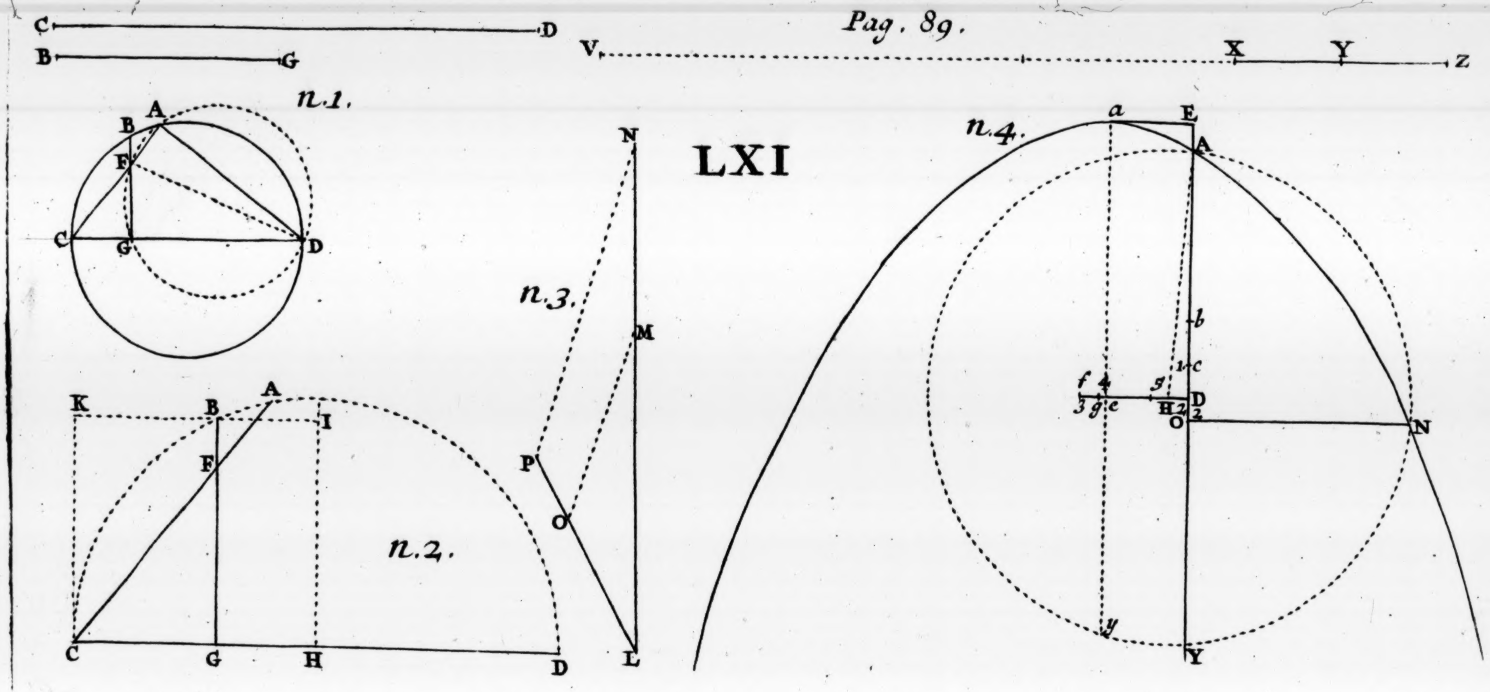
LXI



C D
B G
n.1.
A B F C G D
n.2.
K B A I F C G H D
n.3.
N M P O L
V X Y Z
n.4.
a E A b c f 4 g D 3 g e H 2 O 2 N y Y

so PV $=$ to it, that PX may be $\frac{p^3}{16}$; and lastly P$y = \frac{1}{2}q$, that PZ may be $\frac{pq}{4}$. These being thus prepar'd, if (*n.* 4) A*b* be made $= \frac{1}{2}$ AB (*n.* 1.) *bc* $=$ RT, and *cd* backwards $= \frac{1}{2}$ 𝔓𝔔 we shall light on the point D; and, if we make D*e* $= \frac{1}{2}$ BD, *ef* $=$ PX (*n.* 5. which interval was too big to be represented in the Paper) and from *f* you put backwards *fg* $=$ PZ, and on from *g* beyond to the right hand *gh* $=$ 2RS; we shall light on the point H, *&c.* Which of these two methods is the shortest and fittest for practice, any one, never so little experienc'd, may here see; and first Learners may take notice if they would construct by *Baker*'s form, in the Diagrams *n.* 3. *n.* 5: and the like, they must take care to make the angles 𝔓𝔏𝔑, SPT pretty large; which we have here represented the less to save charges in cutting on Copper.

PROBLEM XIII.

Having given the Diameter of a Circle CD (n. 1. *Fig.* 61.) *and the line BG, which falls on it perpendicularly, (which we have here only rudely delineated) to find the point A from which a right line AC being drawn shall so cut the line BG in F, that AF, FG, CD shall be three continual Proportionals.*

SOLUTION.

If CF be found, the point A will be also had, and the section of the line BG will be made. Make therefore CF $= x$, and (because the perpendicular BG is given, there will also be given the segments of the diameter CG and GD) make GD $= b$, and CG $= c$; then will BG $= \sqrt{bc}$ and CD $= b + c$. Since therefore the $\triangle\,\triangle$ CAD and CGF are right-angled, and have the angle at C common, they will be similar.

Therefore, as CF to CG so CD to CA

$$x \text{ --- } c \text{ --- } b+c \text{ --- } \frac{bc+cc}{x}$$

There-

Therefore AF $= \frac{bc + cc}{x} - x$, and FG $\sqrt{xx - cc}$.

But by the Hypothesis,

as AF to FG so FG to GD

$\frac{bc+cc-xx}{x} - \sqrt{xx\cdot cc} - \sqrt{xx - cc} - b$.

Therefore the rectangle of the extremes will be equal to that of the means,

$\frac{bbc + bcc - bxx}{x} = xx - cc$;

and multiplying by x,

$bbc + bcc - bxx = x^3 - ccx$;

and transposing all to the left,

$x^3 + bxx - ccx - bbc - bcc = o$; i. e. by the *Cartesian* form,

$x^3 + pxx - qx - r = o$.

Therefore (taking b for 1 and L.) the Central Rule will be,

$\frac{L}{2} + \frac{p^2}{8} + \frac{q}{2} =$ AD

and $\frac{p}{4} + \frac{p^3}{16} + \frac{pq}{4} - \frac{r}{2} =$ DH.

or according to our Reduction,

$\frac{b}{2} + \frac{bb}{8} + \frac{cc}{2}$ i. e. $\frac{5b}{8} + \frac{cc}{2} =$ AD

and $\frac{b}{4} + \frac{b^3}{16} + \frac{cc}{4} - \frac{c-cc}{2}$ i. e. $\frac{5b}{16} - \frac{cc}{4} - \frac{c}{2} =$ DH.

Geometrical Construction. Having described upon the given line CD (*n* 2) a semi-circle, and apply'd in it the given perpendicular BG, as the figure shews, you'l have the segments of the Diameter GD $= b$, and to the quantity p in *Baker*'s form, and CG $= c$, which (*n.* 3 where LM $= b$, LO and MN $= c$) will give OP $= cc$ and to the quantity q. Wherefore having describ'd a Parabola (*n.* 4.) and the line VZ $= 2\frac{1}{2}b$, having cut off the fourth part of XZ, and the eighth of YZ (whereof the one will be $= \frac{2\frac{1}{2}b}{8}$, viz. $\frac{5b}{16}$, and

the

the other to $\frac{5b}{8}$) if you transfer A1 = XZ upon the diameter of the Parabola Ay, and moreover 1, 2 or 1 D = to half OP (*n*. 3.) and transversly D3 = YZ and backwards 3, 4 = $\frac{1}{4}$ OP, as also 4, 5 = $\frac{1}{2}$CG (*n*. 2.) you'l have the center H, and having describ'd a Circle at the interval HA, the root NO must be transferr'd from (*n*. 2.) C to F, and continued to A the point sought. In *Baker*'s Form (because the quantity *p* is = *b* or 1) $\frac{p^2}{8}$ is = $\frac{b}{8}$ and $\frac{b^3}{16}$ = $\frac{b}{16}$, and the quantity *q* or *cc* = OP, (*n*. 3.) make therefore in the Diameter of the Parabola A*b* = $\frac{1}{2}$GD, and *bc* = $\frac{1}{8}$GD, (*n* 2.) and lastly *cd* = $\frac{1}{2}$OP (*n*. 3.) and you'l have the point D the same as before. Make moreover D*e* = $\frac{1}{4}$GD and *cf* = $\frac{1}{16}$CG, and then backwards *fg* = $\frac{1}{4}$OP, and lastly *gh* = $\frac{1}{2}$GD, and you'l have the same center H, and the coincidence of the parts in both forms will be pleasant to observe; which otherwise seldom happens.

Other Solutions of the same Problem.

Carolus Renaldinus, from whose Treatise *de Resol. & Compos. Math. Lib.* 2. we have the present Problem, proceeds to solve it in another way, changing it plainly into another Problem: *viz.* he observes, 1. That the angles FAD (see *n*. 1. of our 61. *Fig.*) and FGD, since both are right ones on the same common base FD, are in circle. Hence he infers, 2. (by vertue of the *Coroll.* of the 26. *Prop.* 3. *Eucl.*) that the ▭ ▭ DCG and ACF are equal, and consequently CD, CA, FC and CG are four continued proportionals. Then he observes, 3. That GD is the excess of the first of these proportionals above the fourth CG, and AF is the excess of the second AC above the third CF; and so, since 4. the rectangle of AF and GD is = to the square of the mean proportional FG (for AF, FG, GD, are supposed to be continual proportionals) and this □ FG is the excess, by which the square of the third CF exceeds the square of the fourth CG; now the present Problem will be 5. reduc'd to this other: Having two right lines (CD and CG) given to find two such mean proportionals (AC and FC) that the ▭ of the excess of the first above the fourth (*viz.* of FA into GD) shall be equal to the excess,

by which the square of the third (FC) exceeds the square of the fourth (CG) *viz.* by the square FG.

Wherefore instead of the former he solves this latter Problem, putting for CG, b, for GD, c, so that the first of the given lines CD shall be $= b + c$, and the other GD $= b$; calling the first mean proportional AC, x; and thence denominating the latter $\frac{bb + bc}{x}$ (*viz.* multiplying the fourth by the first, and dividing the product by the second) and moreover he finds the excess of the first $(b + c)$ above the fourth (b) to be c, and the excess of the second (x) above the third $\left(\frac{bb + bc}{x}\right)$ to be $x - \frac{bb + bc}{x}$ i. e. $\frac{xx - bb + bc}{x}$; so that the ▭ of these two excesses is $\frac{cxx - bbc + bcc}{x}$, and because the □ of the third FC is $= \frac{b^4 + 2b^3c + bbcc}{xx}$, having substracted $bb =$ □ GC, there is given the □ FG $= \frac{b^4 + 2b^3c + bbcc}{xx} - bb =$ ▭ of the excesses we just now found. So that now you'l have the Equation

$$\frac{b^4 + 2b^3c + bbcc - bbxx}{xx} = \frac{cxx - bbc + bcc}{x}, \text{ \&c.}$$

We also endeavour'd to find another Solution, by finding an Equation from the line BD (*n.* 1. *Fig.* 61.) as which might be twice obtain'd by means of the two right-angled Triangles FAD and FGD, since it is the hypothenusa of both. But here, besides the former denominations of our Solution we must first give a denomination to the line AD, by making

as CF to FG so CD to AD

$$x - \sqrt{xx - cc} - b + c \text{ so } a - \frac{a\sqrt{xx - cc}}{x}, \text{ \&c.}$$

But whosoever shall prosecute this Solution of ours, or that of *Renaldinus* to the end, will find much more labour and difficulty in either, than in the first we have given.

APPEN-

APPENDIX.

THE Invention of the special Central Rule for the case of *Problem* 1. *Of Cubick Equations*, &c. which may serve for an Example for all the other special ones which belong to our Synopsis, *p.* 354. (and from these special ones) to find a general one.

In *Fig.* 47. *n.* 4. make AD $= b$, DH $= d$; and so we shall have the □ of the radius HA $= bb + dd$. But this □ HA or HN, may be also had otherwise, by putting,

2. For the quantity NO, as sought, the letter x, and by inferring from the known Property of the Parabola,

as L to NO so NO to AO

$$— x — x — \frac{xx}{L}$$

and substracting AD $= b$ from AO, you'l have DO or PN $= \frac{xx}{L} - b$; whose □ is $\frac{x^4}{L^2} - \frac{2bxx}{L} + bb$.

But you also have PH $=$ DH — DP or NO, *i. e.* $d - x$; whose □ is therefore $= dd - 2dx + xx$.

Wherefore adding the □ □ PN and PH, you'l have the □ HN $= \frac{x^4}{L^2} - \frac{2bxx}{L} + xx - 2dx + bb + dd$.

Wherefore the Equation will now be readily had:

$$(\square\,\mathrm{HN}) \text{————————} (\square\,\mathrm{HA})$$

$$\frac{x^4}{L^2} - \frac{2bxx}{L} + xx - 2dx + bb + dd = bb + dd;$$

and taking from both sides $bb + dd$,

$$\frac{x^4}{L^2} - \frac{2bxx}{L} + xx - 2dx = 0.$$

and multiplying all by L^2
and also dividing by x,

$$\begin{array}{l} x^3 * - 2bLx - 2L^2d = 0. \\ \quad + L^2 x \end{array}$$

3. And now farther comparing this Equation with a form like ours in *Probl.* 1. *viz.* with this,

$$x^3 * * - r = 0;$$

It is manifest, since in ours also the third term, *viz.* q is wanting, that the correspondent one to it in the former $-2bL + L^2$ is equivalent to 0; and adding to both sides $2bL$, L^2 will be made $= 2bL$; and dividing by $2L$, $\frac{L^2}{2L}$ will $= b$ or AD. In like manner, since $-r$ in our form answers to the quantity $2L^2d$ in the former; $2L^2d$ will $= r$, and, dividing by $2L^2$, DH or d will $= \frac{r}{2L^2}$: Which is the other Member of the Central Rule to be found.

NB. The Analytick Art has this particularly to be admired in in, that it finds its own Rules by an Analysis: Whereof we have here an evident Example, and several others in the Resolution of affected Quadraticks, *p.* 345. and the following.

II.

The Invention of the Central Rule, in the Case of Fig. 11. *and the like.*

1. $\square$ HA $= bb + dd$ as above.

2. Putting x for NO, as sought, we may infer from a new Property of the Parabola, which we have demonstrated *Prop.*6. *lib.* 2.

as L to NO so OR to AO, *i. e.* (putting a for BA or FO given; that NO $-$ OF *i. e.* NF or OR shall be $= x - a$)

as L to x so $x - a$ to $\frac{xx - ax}{L} =$ AO.

Therefore having substracted AD *i. e.* b from AO, you'l have DO or HP $= \frac{xx - ax - b}{L}$; whose $\square$ is

$x^4 -$

$$\frac{x^4 - 2ax^3 + axx}{L} - \frac{2bxx}{L} + \frac{2abx}{L} + bb.$$

But PN also is = NO — PO or HD, = $x - d$, is = $x^2 - 2dx + dd$.

Therefore having added the □ □ PH and PN, there will come out □ HN

$$\frac{x^4 - 2ax^3 + axx}{L^2} - \frac{2bxx + 2abx}{L} + xx - 2dx + bb + dd,$$

= □ HA *i. i.* $bb + dd$; and taking away from both sides $bb + dd$,

$$\frac{x^4 - 2ax^3 + axx}{L^2} - \frac{2bxx + 2abx}{L} + xx - 2dx = 0;$$

and multiplying every where by L^2 and diving by x,

$$x^3 - 2ax^2 + aax - 2Lbx + 2Lab + 2L^2x - 2L^2d = 0.$$

3. And now comparing this Equation with another form, which shall be like an Equation arising from the Solution of some Problem, *e. g.* with this $x^3 - pxx + qx - r = 0$; to this you'l have = this other,

$$x^3 - 2axx + \left.\begin{matrix} aa \\ -2Lb \\ +L^2 \end{matrix}\right\} x \begin{matrix} +2Lab \\ -2L^2d. \end{matrix}$$

4. Wherefore, because in these equal forms, first, $2a$ is $= p$, a will be $= \frac{p}{2}$ *i. e.* the line BA. Secondly, Because aa (or $\frac{p^2}{4}$) $+ L^2 - 2Lb = q$;

Therefore $2Lb = \frac{p^2}{4} + L^2 - q$, and dividing by $2L$,

$$b = \frac{L^2}{2L} \text{ (or } \frac{L}{2}\text{)} + \frac{p^2}{8L} - \frac{q}{2L} = \text{AD}.$$

Thirdly, Because $2abL - 2dL^2 = r$ *i. e.* $2abL - r = 2dL^2$; therefore dividing by $2L^2$, d will be $= \frac{2abL}{2L^2} - \frac{r}{2L^2}$ i. e.

 $d =$

$d = \frac{ab}{L} - \frac{r}{2L^2}$ *i. e.* resolving $\frac{ab}{L}$ into equivalent Terms expressed by p and q,

$d = \frac{p}{4} + \frac{p^3}{16L^2} - \frac{pq}{4L^2} - \frac{r}{2L^2}$; which is the other member of the Central Rule to be found.

viz. a is $= \frac{p}{2}$; $b = \frac{L}{2} + \frac{p^3}{8L} - \frac{q}{2L}$

Therefore ab will be $= \frac{Lp}{4} + \frac{p^3}{16L} - \frac{pq}{4L}$

Therefore $\frac{ab}{L} = \frac{p}{4} + \frac{p^3}{16L^2} - \frac{pq}{4L^2}$

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